HIGHER SURVEYING

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PREFACE

It has frequently been pointed out to the Author that his practical treatise, " The Field Manual ", was lacking in bias towards the immediate demands of the various written examinations in surveying and geodesy; and that it was hoped that he would supplement this in some way with a selection of examples and questions from the purely tutorial point of view of meeting the requirements peculiar to degrees in engineering in the various universities.

It is hoped that the present work will fulfil these requirements without digressing too far from the field aspects of an essentially practical subject. The art of surveying is not acquired from text-books or in tutorial classes; and true appreciation of problems arises only from data brought in from the field.

The text consists of summaries taken from the author's lectures in surveying and geodesy at Queen Mary College, and the examples and questions are selected mainly from papers set by him in the various examinations of the University of London, and by the kind permission of the Senate they are here reprinted. As implied by " higher surveying ", it is assumed that the student is acquainted with the elements of surveying and setting-out, and thus it is possible to preclude such operations as chain surveying, levelling, traversing, and plane tabling. On the other hand, the desirability of recapitulation is recognised by the insertion of certain problems of an intermediate nature, which are introduced by concise summaries in place of the fuller treatment accorded to matter of advanced standard.

Problems for solution by the student have been arranged at the ends of the various sections; and those from the examinations of the University of London (U.L.) are marked by an asterisk * when of Part I standard, and a dagger † when of Part II, or final, standard; the double dagger †† denoting questions to which one hour may be allocated. This convention is omitted from the additional problems, which are taken from recent examinations of the following authorities, since the organisation and treatment of the subject varies considerably in different university colleges. The author expresses here his thanks to the Senates, Courts, or Councils of the following universities, colleges, and institutions for their kind permission to reprint the questions selected:

University of Birmingham (U.B.);
University of Cape Town (U.C.T.);
University of Dublin (U.D.);
University of Glasgow (U.G.);
Thomason College of Civil Engineering, Roorkie, India (T.C.C.E.);
Institution of Civil Engineers (I.C.E.)

The so-called renaissance in the design of surveying instruments is discussed in the introductory section, and the developments of aerial methods at a later stage; and the author would like to express his indebtedness to the following makers by whose courtesy the illustrations are reproduced: Messrs. Cooke, Troughton & Simms; Messrs. C. F. Casella & Co.; Messrs. W. F. Stanley & Co.; Messrs. E. R. Watts & Son; and the Williamson Manufacturing Coy.

Particular mention is made of the courteous help given by the Syndics of the Cambridge University Press in permitting the author to reproduce a number of diagrams from his book on photographic surveying, including the schematic plan of the stereoautograph.

The author also takes this opportunity of expressing his acknowledgments for much information derived from various sources named in the text; his thanks to Mr. A. J. V. Gale, M.A., for his helpful advice while the book was in the press, and to Mr. A. N. Utting, of the Cambridge University Engineering Laboratory, for preparing most of the drawings from which the diagrams are reproduced.

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SECTION I

INTRODUCTION

Although it is assumed that the student has some knowledge of the construction and use of surveying instruments, particularly the level and theodolite, he may not be acquainted with the more recent improvements and developments. Consequently greater emphasis is laid upon the discussion of these patterns, the main features of which are summarised in the following note.

Modern instruments. Since the end of the War of 1914–18 the changes in the design of surveying instruments have been so radical that the title of these observations is by no means unqualified.

The chief innovation has been compactness of design, giving portability and, above all, simplicity of packing, the metal carrying case being an outstanding improvement. Contributory to this are the internal-focussing telescope, precision workmanship, and selection of materials to afford lightness without sacrifice of strength.

Another feature is not merely the subordination of verniers to micrometer microscopes on small instruments, but the introduction of optical devices so that opposite portions of the divided circles can be read simultaneously, thus avoiding the disturbances arising from movements around the tripod. An outstanding development is the precise reversible bubble, which has led to simplification in the adjustment of levels.

Whether all these constructional and operational innovations are improvements is a matter to be decided by the demands upon the instrument: if for geodetic and precision work or for setting-out and land surveying. For example, the " niveau de pente ", the improved principle of the old builders' level of centering the bubble for each sight, is most desirable on line work, whereas it is by no means amenable to trial and error work such as driving slope stakes or locating points on actual contours. Also optical means of reading circles in lower grade surveys or setting-out is by no means advantageous on account of the incompatibility between the degrees of precision of linear and angular measurement.

In geodetic and precise work the modern instrument is an acquisition, yet, traditionally or otherwise, there is still a demand for the older patterns, in heavy work and, particularly, overseas, where open means of

adjustment and improvised repair are desirable. It is therefore essential in a book of this nature that due consideration be given to the earlier models of surveying instruments, even though these are obsolete in so far as recent catalogues are concerned. The study of the subject must embody a historical note, tracing the evolution and development of essential principles. Also it is necessary to recognise the " lag " concomitant with instruction from the equipment available, not only in colleges and institutions, but also in many engineers' offices.

ARTICLE 1 : TELESCOPES

Two general types of telescopes are embodied in levels, theodolites, etc. : draw tube (or external-focussing) and internal-focussing.

Draw tube type. This type, which is usually found in older pattern instruments, is the Huygenian type of Kepler's telescope. It consists essentially of the following parts : (*a*) Two body tubes, one of which is moved within the other by means of a rack and pinion, operated with a focussing screw ; (*b*) the objective, comprised of an achromatic combination of a hard crown double-convex glass (outer) and a dense flint concavo-convex glass (inner) ; (*c*) the diaphragm, which is " webbed " with spider lines, lines on glass, or metal points ; and (*d*) the Ramsden eyepiece, which consists of two similar plano-convex lenses, spaced two-thirds the common focal length apart, with the convex faces inwards. Incidental fittings consist of stops for cutting off extraneous rays, and a sun cap or ray shade fitting over the objective. Formerly the eyepiece end moved in focussing, but for better balance the objective is moved in later patterns.

The chief objections to this type are the greater length for a given power, and the defects arising from construction. In the author's opinion there has been a tendency to exaggerate the latter objection. A special form of draw tube telescope is **Porro's anallatic telescope.**

The simple theory of the external-focussing telescope is too well known to require recapitulation.

Internal-focussing type. Characteristic of modern instruments is the type in which the diaphragm is at a fixed distance from the objective, the draw tube being superseded by an additional double-concave focussing lens, which is moved by means of the focussing screw. Otherwise it is similar to the preceding pattern. The merits and defects are concisely as follows : (1) Dust, etc., do not enter the telescope, and in some climates

the effects of damp are negligible. Against these is the fact—"More glass, less light". (2) Balance of telescope is practically unaffected by focussing, and the errors arising from faulty construction are not so serious as in draw tube focussing. The magnifying power is not constant. (3) A simple anallatic telescope can be approximated to closely by judicious choice of dimensions, though earlier patterns were at fault in this regard. On the other hand, certain advantages of the true anallatic telescope are lost, particularly the adjustment to exact multiplier, normally 100. In general, however, the advantages outweigh the disadvantages, though not always to the extent usually represented. Another type of internal-focussing telescope is that introduced in the Zeiss or Wild level (p. 10).

The optical size of a telescope is determined by the focal length f of the objective, which is an indirect dimension in the internal-focussing type; and the magnifying power m is expressed by the ratio f/f_0, where f_0 is the focal length of the eyepiece. Usually m is from $\times 20$ to $\times 30$ (diameters).

The focal length of the Ramsden eyepiece is expressed by the relation $\dfrac{f^2}{2f-a}$, which reduces to $\frac{3}{4}f$, where f is the common focal length of the glasses and $a = \frac{2}{3}f$ the distance between them. The eyepieces of Continental telescopes are frequently provided with a diopter scale.

Erecting eyepieces are frequently supplied, though more particularly in American instruments or those constructed for colonial use. The optical arrangement of these eyepieces consists of (*a*) an object lens, (*b*) an amplifying lens, (*c*) a field lens, and (*d*) an eye lens. Usually the loss of light through the two additional lenses is a more serious objection than the inconvenience arising from inversion with the Ramsden eyepiece.

Internal-focussing telescope. Let l be the fixed distance between the objective and the diaphragm, and d the distance between the objective and the negative focussing lens, the focal lengths of which are f and f' respectively (Fig. 1).

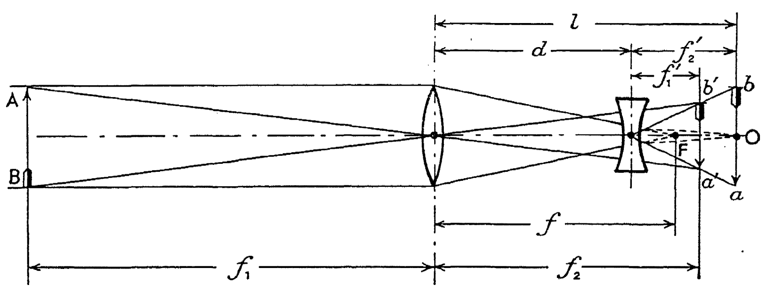


FIG. 1.

Then for a virtual focus $a'b'$ at distance f_2 from the objective, the negative lens being absent,

$$\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{f}, \quad \text{or} \quad f_2=\frac{ff_1}{f_1-f}. \quad \text{.......................(1)}$$

After refraction through the negative lens, the actual image ab is formed on the diaphragm at a distance f_2' behind the negative lens ; then

$$\frac{1}{f'}=\frac{1}{f_1'}-\frac{1}{f_2'}=\frac{1}{f_2-d}-\frac{1}{l-d}, \quad \text{and} \quad f_2=\frac{ld+lf'-d^2}{l-d+f'}, \quad \text{........(2)}$$

leading to $$d=\tfrac{1}{2}(l+f_2)\mp\sqrt{\tfrac{1}{4}(l-f_2)^2+f'(l-f_2)}. \quad \text{.....................(3)}$$

Equations (2) and (3) are the more convenient forms for practical calculation, for the final expression on substituting for f_2 from (1) is a complex quadratic in d.

Many problems can be solved from the simple conjugate relations. Usually the magnifying power of an internal-focussing telescope is expressed as

$$m=\frac{ff'}{f_0(f-f'-d)},$$

where f, f', and f_0 are respectively the focal lengths of the objective, the focussing lens, and the eyepiece, d being the distance between the objective and focussing lens for infinity focus.

The magnifying power $m=\dfrac{f_e}{f_0}$, where f_e is the equivalent focal length and f_0 the focal length of the eyepiece. For infinity focus $f_2=f$ in (1), and it is evident in Fig. 1 that

$$f_e=\frac{f_2'}{f_1'}\cdot f_2=\frac{l-d}{f-d}\cdot f,$$

f_2 becoming f with the lens removed.

$$m=\frac{l-d}{f-d}\cdot\frac{f}{f_0}, \quad \text{....................(4)}$$

The more usual form,

$$m=\frac{ff'}{(f-d-f')f_0},$$

is found by substituting for l from (2) in (4) with $f_2=f$. The foregoing treatment is extended to the internal-focussing tacheometer in Art. 7 (p. 64).

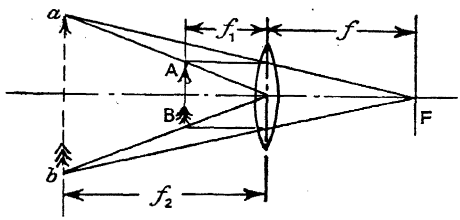


FIG. 2.

Microscopes. The relative size of the object AB and its image ab as viewed through a microscope is determined by the ratio of the conjugate foci; namely, $\frac{AB}{ab}=\frac{f_1}{f_2}$ (Fig. 2). If f be the focal length of the objective lens, it follows from $\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}$ that when $f_1=f_2$, the object and image are the same size and the object is at a distance $2f$ from the lens. When f_1 is less than $2f$, the image is greater than the object, and is *real* until $f_1=f$; but when f_1 is less than f, the image will be *virtual*, erect, and magnified, as in the figure.

Thus, in the case of micrometer microscopes (p. 35), if the imaged graduations are too large, f_1 must be increased, since the ratio becomes

$$\frac{ab}{AB}=\frac{f}{f-f_1} \text{ (with } f_2 \text{ negative).}$$

QUESTIONS ON ARTICLE I

1†. State concisely the merits and defects of internal-focussing telescopes.

A telescope of the above type is being designed to the following specification:

Magnification × 30, focal length of objective $7\frac{1}{2}''$, focal length of eyepiece $\frac{1}{2}''$, with distance between objective and focussing lens at infinity focus, $6''$.

Determine the focal length of the negative focussing lens and the fixed distance between the objective and diaphragm. (U.L.)

$[l=9''\,;\ f'=3'']$

2†. The following data refer to an internal-focussing telescope:

Total length from centre of objective	$8''$
Focal length of convex object glass	$7''$
Focal length of concave focussing lens	$5{\cdot}625''$
Total movement of concave focussing lens	$1''$

Assuming that one limit of travel of the internal lens corresponds to the position for infinity focus of the telescope, determine the shortest sight that can be focussed.

[6·71 ft.]

3*. Submit a neat sketch showing the longitudinal section of a telescope which is focussed by means of a draw tube. Add a brief note on the optical theory involved.

4*. Describe how you would make a field determination of the magnifying power of a surveying telescope. Explain concisely the following terms: (*a*) Achromatism, (*b*) Aplanatism, (*c*) Definition, (*d*) Illumination, and (*e*) Field.

5†. Show that the magnifying power of an internally-focussing telescope is $m = \frac{ff'}{f_0(f - f' - d)}$, where f, f', and f_0 are respectively the focal lengths of the objective, the focussing lens, and the eyepiece, d being the distance between the objective and focussing lens for infinity focus. (U.L.)

ARTICLE 2: LEVELS

Levels may be divided into three categories: namely, (1) dumpy levels, (2) " Y ", and (3) other reversible levels, the last class including precise levels.

All levels consist essentially of the following four components: (*a*) the **telescope,** the diaphragm of which is usually "webbed" with one horizontal line and two vertical lines, a stadia interval being sometimes added —often to the confusion of a student's readings; (*b*) the **spirit level,** which consists of a glass tube, or phial, usually curved internally and so filled with spirit so that it contains the essential bubble of vapour; (*c*) the **limb,** which carries the telescope and attached level tube, connecting these to (*d*) the **levelling head,** the *tribrach* (or parallel plate) device, by which it is attached to the tripod.

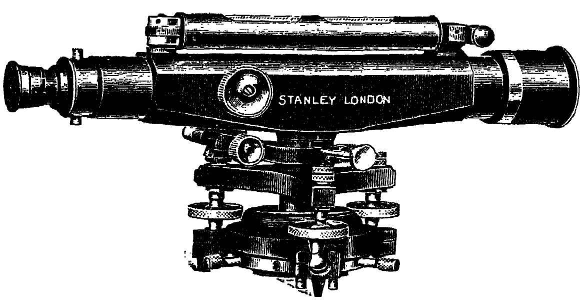


FIG. 3. Stanley's " Engineers' " Dumpy Level.

It may be observed that the spirit level introduces the principle of gravitational levelling, the longitudinal radius of curvature of the bubble tube R being the " equivalent plumb line ", say 250 ft., which suggests a high degree of *sensitiveness* in the physical sense. Also the divisions of the bubble tube, often figured, as, for example, 20″ per division, provide a precise means of evaluating small vertical angles, the sensitiveness per second of arc being $\frac{R}{206265}$, and the angular value of a division $v/R \sin 1''$ with v the length of a division in the unit of R.

In operation, the level may be said to embody three fundamental lines : (*a*) the **bubble line**, an imaginary tangent to the bubble (cf. " bubble axis ") ; (*b*) the **line of collimation**, usually defined as the line between the centres of the objective and diaphragm ; and (*c*) the **vertical axis**, the axis of rotation in the horizontal plane in the case of " traversing " levels. The **conditions of adjustment** are that (i) the line of collimation must be parallel to the bubble line, and (ii) the bubble line must be perpendicular to the vertical axis (while it might be added (iii) that the line of collimation should coincide with the axis of the telescope).

Condition (i) alone is the condition of accurate levelling ; condition (ii) is a convenience, not a necessity, and is not involved in levels provided with a tilting screw, which is a desirable feature in line work and precise levelling, though " traversing " is a great advantage in certain operations. Condition (iii) normally follows in the case of reversible levels. Establishing the correct relations between the fundamental lines is known as effecting the **permanent adjustments**: an operation once viewed with diffidence by surveyors, but now becoming a mere detail of the work (see Art. 3).

Dumpy levels. Ever since Gravatt introduced this form in 1848, it has been the most popular level among British engineers, its traditional simplicity and stability commending its use. The main objection against the type was that its adjustment involved the " two-peg " test, which fundamentally is the only absolute field test of any level, and actually involves little experience in carrying out. The fixed telescope in itself is the best guarantee of constant adjustment ; and, in consequence, there will always be a field for the dumpy principle, thus characterised. A recent Zeiss model for work of high order is fundamentally a dumpy level provided with internal and external bubble reading, wedge-lined diaphragm, and, if desired, the rocking prism or optical micrometer.

Wye levels. For several years many American and Continental surveyors have adhered to this instrument, the original form of which was introduced by Sissons in the eighteenth century. In Y levels the telescope is removable and reversible end for end in the Y's, being secured in use by means of metal straps. Rotation about the geometrical (and optical) axis of the telescope was the maker's ideal—collimating, as it is called—

and this, combined with end for end reversals, led to an easy process of adjustment. The various open and loose parts were a serious drawback in engineering surveys, yet the modern patterns are still used considerably, particularly in America. The **Kern level** was a well-known model of this class.

Reversible levels. Among the first instruments representative of this class may be cited the well-known instruments both of Cushing and Cooke, which retained the collimating principle of the Y level and were claimed to embody the qualities of the dumpy level. In 1910, Heinrich Wild, a Swiss engineer, developed the well-known Zeiss level: then an innovation, which has doubtless influenced the design and finish of instruments since 1918, justifying the term "Modern Instruments". Characteristic of these are compactness, radical changes in materials of construction and finish, facility of adjustment, and extended use of the niveau de pente principle, by which the bubble is centralised for each reading of the level.

The original Zeiss level merited its place in the more precise work, but many engineers of experience preferred the stability inherent in the modern versions of the older types, the easy access of water to the prisms

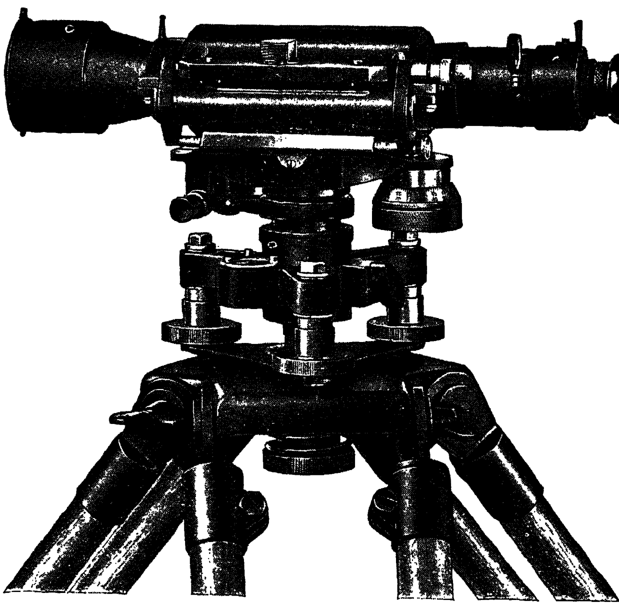


FIG. 4. Reversible Level.
(Cooke, Troughton & Simms.)

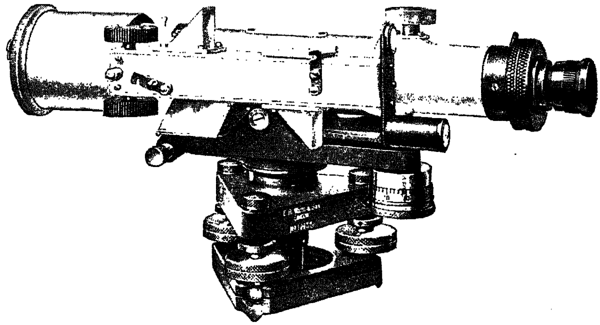


FIG. 5. Watts' Self-adjusting Level.

impairing the use of the instrument in wet climates. The modern reversible levels of Messrs. Cooke, Troughton & Simms, Messrs. E. R. Watts & Son, and Messrs. W. F. Stanley & Co., introduce many interesting features, particularly in the facility of adjustment by virtue of precise construction, combined with the use of special bubbles and reading devices.

Improved level tubes. Great advances have been made in recent years in the design, grinding and mounting of the bubble tubes, or vials. In most modern levels the bubble is attached to the side of the telescope, normally on the left. The bubble mirror has been developed, as in certain Watts' levels, the mirror folding down to protect the vial. Prism reading is embodied in many levels, the " constant " bubble being read with a simple prism, and the reversion bubble with a double-prism device, the coincidence of the ends being viewed in a 45° prism, as in certain Cooke and Zeiss levels.

Among the improved bubbles are **Cooke's reversion level**, which has similar scales on opposite sides of the vial, these divisions being in perfect register with each other so that for the horizontal condition the bubble comes to rest centrally with respect to the divisions in either position. This feature is the basis of the simplified adjustment of the **improved reversible level**, the opposite scales being in register with each other within 2″ of arc. The " constant " bubble of Messrs. Watts & Son has the property that its length does not vary with changes of temperature : a fact evinced by the N.P.L. certificates, which cover a test between 0° and 130° F. The cross-section of this vial is approximately elliptical, and the volumes of air and spirit are so proportioned that the decrease of surface tension with the rise in temperature counterbalances the expansion of the spirit, rendering the length of the bubble constant over a range of upwards of 100° F. Thus it is necessary to observe only one end of the bubble, either through a reading lens or a simple prism. The bubble is

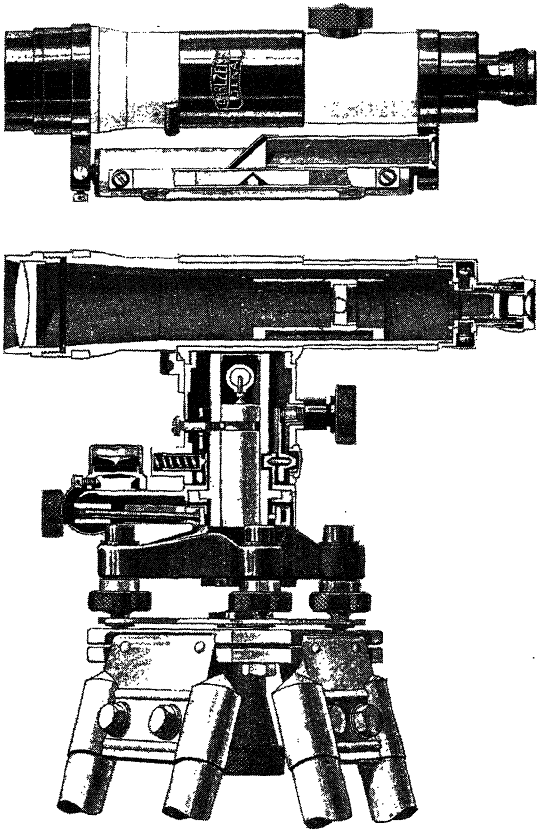


FIG. 6. Zeiss Reversible Level.

reversible in the **Watts' self-adjusting level**, leading to an expeditious test for the all-important collimation adjustment.

A feature of those levels which utilise the niveau de pente principle is the circular or bull's-eye vial used in the preliminary levelling up of the instrument. The grinding ensures a perfectly spherical surface, while the sealing prevents evaporation or leakage of the spirit. Circular bubbles have to some extent supplemented the short plate levels on theodolites.

Zeiss level. The telescope of the best-known reversible model is essentially of the internal-focussing order, but unique in that it embodies *two* achromatic, plano-convex objectives, similar, and of the same focal length, the diaphragms consisting of lines engraved on the inner plane surfaces of both lens combinations. When the normal objective is used,

as in ordinary levelling, the eyepiece, which is figured in diopters, is focussed by screwing it into its tube ; and when the subnormal objective is used, as in testing the permanent adjustment, or in precise levelling, an annular cap is placed over the normal objective, and the eyepiece is inserted in this. Otherwise the negative focussing lens is typical of internal focussing, and the theory of the telescope is that described with respect to internal-focussing telescopes on p. 3.

The telescope can be rotated through 180° in an outer tubular body, which is provided with lugged collars, the projections of which carry the level tube, the prism case and mirror ; normally on the left. It therefore follows that staff readings can be taken in four positions : 1, *normal* (bubble left, erect) ; 2, *inverted* (bubble right, inverted) ; 3, *reversed* (bubble left, inverted) ; 4, *subnormal* (bubble left, erect).

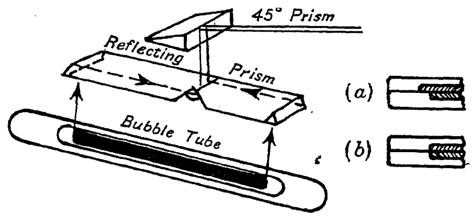


FIG. 7.

A reversion level tube is carried between the lugs of the collars, one end being flexible-jointed, and the other fitted with capstan screws. Only when the bubble error is large are these screws used, but an apparent adjustment is made by moving the prism case longitudinally by means of a milled screw, which is locked with a circular nut. Underneath the bubble tube is a long mirror, which can be rotated to illuminate the bubble ; above the bubble tube is the prism case which encloses the reflecting prism and carries on top the reading prism, the arrangement being as in Fig. 7, where (*a*) and (*b*) show the images of the ends of the bubble for the displaced and rectified positions.

The levelling head consists of a pillar, which terminates at the top in an interrupted screw thread for attachment to the telescope body, and at the bottom in a collar surrounding the outer sleeve of the vertical axis. The pillar acts as a crank which moves the telescope and its attachments about a horizontal axis, movement being effected by means of a tilting screw in the outer casing. The steel vertical axis rises from the upper plate of the levelling head, and, being cylindrical, affords no support, but merely guides the rotatory motion of the telescope. The construction of these parts involves some interesting features, as also does the slow-motion device. A circular spirit bubble is attached to the outer body, as the means of preparatory adjustment by the foot screws.

The level is attached by a capped screw which passes through the tripod head into the lower plate of the levelling head.

Later models were provided with adequate protection of the bubble and fine adjustments. The recent model is fitted with a microscope for reading the bubble coincidences; and the adjustment is effected from two positions of the telescope by virtue of exact parallelism of the two bubble tangents.

Cooke's improved reversible level. This instrument, a development of the Improved Engineers' Level, introduces a reversion level of so accurate a type that, in conjunction with precision construction, only the *normal* and *inverted* positions of a normal telescope (1 and 2) are concerned in adjustment tests. The bubble is viewed from the eye end of the telescope in a compound speculum mirror. A differential tilting screw is provided for setting the bubble central at each observation. The telescope is internal-focussing with a glass stadia diaphragm in an interchangeable cell, the ray shade being provided with cross-sighting slits. This model has been superseded, but the reversible feature is now embodied in the **Geodetic Level**, as designed specifically for primary levelling, coincidence setting of the bubble being obtained through the eyepiece.

Watts' self-adjusting levels. Characteristic of these instruments is the constant bubble of the reversion type and the simple prism device, which is duplicated for observing one end only of the inverted and normal bubble. This instrument is made in the 11″ and 14″ and 21″ patterns, all of which are fitted with a micrometer tilting screw, approximate levelling by the tribrach being controlled by a circular bubble. The first two patterns are internal-focussing with respective equivalent foci of 12″ and $15\frac{3}{4}$″, but the largest model is external-focussing, the eye end extending. Glass stadia diaphragms are fitted to all models, and the optical micrometer can be supplied (Fig. 5).

Levelling staves. Staves are of two kinds: **self-reading** and **target** patterns. The former are read from the level and the latter on the vernier of a target which is sighted to position on the staff. Some target staves are also provided with "speaking" divisions, as in the well-known Philadelphia and New York rods. Self-reading staves are preponderant in the United Kingdom, the well-known **Sopwith ladder** being the division most commonly used. There is, however, a tendency to introduce the target staff to precise work, and, incidentally, this form is an acquisition in reciprocal levelling and in adjusting instruments.

For precise work an invar strip is sometimes inset in the staff. In one pattern the strip is divided at each 0·02 ft., and this slides in the staff, the lower end of the strip being shod with steel. The staff may be provided with a plummet or a staff bubble to ensure that it is held truly vertical. This pattern is used in conjunction with the plate glass micrometer (p. 13).

*Example**. Outline the theory of the parallel-plate micrometer as used in precise levelling.

*Example**. A parallel-plate micrometer attached to a level is to show a displacement of 0·01 ft. when rotated through 15° on either side of the vertical. Calculate the thickness of the glass when the refractive index is 1·6. State also the staff reading to the nearest thousandth of a foot when the micrometer is brought to division " 7 " in sighting the next lower reading of 4·24, the divisions running 0 to 20 with 10 for the normal position. (U.L.)

Parallel-plate micrometer.—This device, which facilitates the reading of an ordinary staff to 1/1000 ft., consists of a parallel glass plate fitted over the objective and given a tilting motion by the rotation of a micrometer head. This head is divided into 20 parts, or 10 for each direction of tilt, each division giving the reading to 1/1000 ft. When the divisions run 0–10–0, the reading is 10 when the plate is vertical, and the increment is added or subtracted according as the next lower or next higher 0·01 ft. is read, the plate being tilted in opposite directions. If, however, the division is 0 to 20 with 10 for the vertical position, the increment is always added if the drum is set to zero at the outset, and the horizontal hair, as imaged, is traversed back to the next lower 0·01 on the staff.

Let t be the thickness of the glass and μ the refractive index from air to glass (Fig. 8).

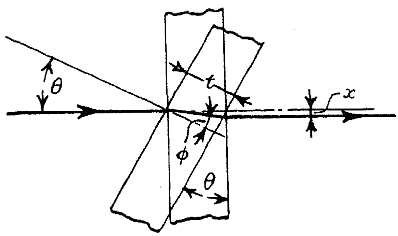


FIG. 8.

$$t(\tan\theta-\tan\phi)\cos\theta=t\sin\theta\left\{1-\frac{\cos\theta}{\sin\theta}\,\frac{\sin\phi}{\sqrt{1-\sin^2\phi}}\right\}$$

$$t\sin\theta\left(1-\frac{\cos\theta}{\sin\theta}\,\frac{\sin\theta/\mu}{\sqrt{1-\dfrac{\sin^2\theta}{\mu}}}\right)$$

$$=t\sin\theta\left(1-\frac{\cos\theta}{\sqrt{\mu^2-\sin^2\theta}}\right)=t\sin\theta\left(1-\frac{\sqrt{1-\sin^2\theta}}{\sqrt{\mu^2-\sin^2\theta}}\right).$$

If θ be small, $\sin\theta=\theta$, and $\sin^2\theta$ is negligible ; then $x=t\theta\left(1-\dfrac{1}{\mu}\right)$.

[Thickness, 0·0995″ ; reading, 4·247.]

QUESTIONS ON ARTICLE 2

1*. Describe the essential differences between the dumpy level and Cooke's new reversible level, and state clearly your reasons for selecting one of these types in preference to the other. (U.L.)

2*. The salient features of modern levels may be said to consist of the following: (*a*) internal focussing; (*b*) bubble reading devices; (*c*) simplified adjustment; and (*d*) levelling up independently of the foot screws.

Discuss the merits and demerits of these features from the engineering standpoint. (U.L.)

3*. State concisely why variations in the length of a spirit bubble are objectionable, and explain how a bubble tube can be compensated so that the length of the bubble is independent of the temperature.

Describe how you would make a field determination of the radius of a bubble tube, and indicate how you would deduce therefrom the value of the sensitiveness. (U.L.)

4*. Assuming that levels may be grouped into (1) old, (2) improved, and (3) modern instruments, describe concisely the essential differences between an instrument representative of each category.

[Additional notes on levelling instruments will be found in Article 3.]

5*. Describe with reference to neat sketches any *one* of the following types of reversible levels: (*a*) Zeiss pattern; (*b*) Cooke's improved reversible level; (*c*) Watts' self-adjusting level.

6. A tilting type level, fitted with a graduated drum, is accurately adjusted so that the drum reads 0·00 when the sensitive bubble is central. It is required to set out an up gradient of 1/160; what will be the reading set on the drum?

Using the same instrument to obtain distance, a staff was held vertically at a point B, and readings were taken with the level from a point A, the subtense length on the staff being 5 feet. If the drum readings were 2·35 and 5·85, find the distance between A and B in feet.

Explain the principle of the special parallel plate attachment, which may be fitted to a modern level so that the readings on a staff may be taken to 0·001 feet. Describe the Constant Bubble which is fitted to many modern levels. (U.B.)

[If a complete turn of micrometer drum corresponds to 1 in 1000, 1/160 requires 6·25 turns. Likewise distance $AB = 1429$ ft.]

7. Make a neat sketch of a level of the Zeiss type, and explain clearly its use and advantages. (U.D.)

8. Explain what is meant by the sensitiveness of a level tube. Describe how you would determine in the field the sensitiveness of a tube attached to a dumpy level. (U.D.)

9. Make sketches to illustrate the essential features of any pattern of tilting level, i.e. one in which the line of sight and bubble tube axis are placed horizontal by means of a fine levelling screw.

Describe the collimation adjustment of the instrument. (I.C.E.)

10. Explain the following :

Optical centre of a lens, level surface, geoid, old Ordnance Datum, new Ordnance Datum.

Show clearly why it is necessary for accurate levelling to have the same horizontal distance between a back sight reading and its foresight reading.

How may modern levels be employed to obtain distances by the subtense method? (U.B.)

11. (*a*) Describe in detail how you would determine the difference in level between two points on opposite banks of a river with a dumpy level in which the line of sight is not parallel to the bubble axis, and the bubble axis is not perpendicular to the vertical axis.

Equipment for adjusting the level is not available. The errors in adjustment may be taken as small.

(*b*) A reading is taken on a levelling staff at a distance of 400 feet with the bubble central.

If the telescope is tilted until the reading has changed by 0·1 ft., the bubble has moved through four divisions.

What is the sensitivity of the bubble? (U.C.T.)

[12·9″ per div.]

ARTICLE 3: ADJUSTMENT OF LEVELS

The conditions of adjustment of any level are :

(i) Fundamentally, the line of collimation shall be parallel to the bubble line.

(ii) Constructionally, the line of collimation should lie in the axis of the telescope, the condition being desirable in the case of the dumpy level and essential in Y and all reversible levels.

(iii) Conveniently, the bubble line should be perpendicular to the vertical axis in order that the bubble traverses, or remains central for all directions of the telescope.

It is desirable that the present article should be comprehensive, since numerous instruments of the older patterns are still in use and various types of these are still manufactured. Accordingly, there are four classes of instruments as far as their permanent adjustments are concerned :

(1) Dumpy levels ; (2) Y levels ; (3) reversible levels ; and (4) modern reversible levels.

(1) **Dumpy levels** may be said to be of the (*a*) *single*, (*b*) *double*, and (*c*) *treble* adjustment types, according as these (modern, improved, or old patterns) have one, two, or three means of effecting the adjustments. In (*a*) there are the level tube screws only ; in (*b*) the diaphragm screws and either the level tube screws or the limb screws ; and in (*c*) the diaphragm screws and both level tube and limb screws. The all-important parallelism

of the bubble line and line of collimation is established in (*a*) alone, such being quickset (non-traversing) levels ; in (*b*) the mutual perpendicularity of the bubble line and vertical axis may also be established ; while in (*c*) the choice of limb or level tube screws is open to effect this perpendicularity, the use of limb screws leaving the diaphragm screws for the mooted process of making the line of collimation coincident with the axis of the telescope (**Gravatt's three-peg test**). The reduction of open means of adjustment emphasises the development of precise workmanship, the maker, like the surveyor himself, formerly having to resort to external means of adjustment. Likewise the steps peculiar to the adjustment of the Y level are now inherent in the precise construction of the modern reversible level.

Characteristic of the dumpy level is the **two-peg test,** by which the parallelism of the bubble line and line of collimation is tested or verified. This test is based upon the fact that if these lines are not parallel, the bubble line being horizontal, then the line of collimation on rotation about the vertical axis will generate a flat vertical cone, and, at equal distances from the instrument, the base of this cone will determine staff readings, the difference of which is the true difference of level. Also the error in collimation will be a maximum when the greatest inequality exists between the lengths of the sights.

In order to avoid trial and error, the instrument is first set up midway between pegs A and B, and then over A or B, or, preferably, at C, a convenient distance behind A or B in the line AB produced. The term " peg " merely suggests the necessity for a firm footing for the staff, and the experienced surveyor usually establishes the marks A, B, and C on a hard surface near his work, in order to test his level from time to time. Old text-books suggest pegs driven flush with the surface of a pond, and, in the absence of a chain, the method of reciprocal levels may be used. The test is the only absolute field verification of the accuracy of a level, and in this connection a Y level might respond to its own tests and yet be inaccurate on account of defective collars, etc.

(2) **Y levels,** like reversible levels, are " collimated " first by rotating the telescope until the centre of the cross-hairs remains on a fixed staff reading, or mark, or an illuminated point if the adjustment is carried out indoors. Next the bubble line is set parallel to the line of collimation by reversing the telescope end for end in the Y's, and adjusting the level tube until the bubble is central for both the normal and reversed positions. Finally, the axis is made truly vertical, making the bubble traverse, by conjoint use of the Y nuts and foot screws (Fig. 11).

(3) **Reversible levels,** such as the well-known instruments of Cooke and Cushing, are also provided with three means of adjustment, as in the case of Y levels : (*a*) level tube screws, (*b*) diaphragm screws, and (*c*) limb nuts. As in the case of the Y level, the telescope is collimated first, the

adjustment being effected likewise. Next the line of collimation is set perpendicular to the vertical axis by means of the limb nuts after a mid-point has been established by sights with the telescope normal and then reversed end for end in its socket (or by interchange of objective and eye-piece). Finally, the axis is made truly vertical, making the bubble traverse by conjoint use of the level tube screws and the levelling screws.

(4) **Modern reversible levels** are adjusted by a sighting test similar to the second adjustment of the preceding instruments, the Zeiss pattern involving four sights or readings against two normally in the other patterns. In effect, the bubble line is set parallel to the horizontal line of collimation thus established, the level tube screws (or their equivalent) being used.

In a book of this nature much of the technique of manipulation and adjustment is necessarily omitted, and the following examples are intended to emphasise the essentials so often missed in cursory descriptions. Consequently the use of tribrach or plate screws with reference to the bubble is assumed to be understood ; and the expression " half and half adjustment of the bubble " implies that the level tube is over a tribrach screw (or parallel to two screws), or is over one pair of plate screws, and that the bubble is centralised, half with its adjusting screws and half with the foot screws beneath the bubble. The bubble line is thus set perpendicular to the vertical axis and this axis made truly vertical in the plane of the telescope, and the axis is then set truly vertical in a vertical plane at right angles by sole use of the relevant foot screw or screws.

Example. Detail concisely how you would test and, if necessary, adjust the following types of dumpy levels : (A) improved double adjustment (1904) ; (B) old treble adjustment (1884) ; (C) modern single adjustment (1928).

General. Set up the level firmly at L (Fig. 9) on fairly level ground in a situation such that sights LA, LB, of at least $2\frac{1}{2}$ chains, can be taken in the same straight line, on either side of the instrument, noting that the tribrach sprang (or the lower parallel plate) is fairly horizontal, and (except with quickset types) that one tribrach screw (or a pair of plate screws) is beneath the telescope when sighting along AB. Measure equal distances of (say) 2 chains and drive pegs firmly at A and B. Also measure 1/10 AB (say) 40 links in AB produced, to C, which preferably should be behind the peg of lower reading.

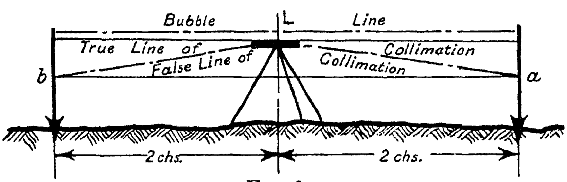


FIG. 9.

(A) **Improved double adjustment.** (1) Level up the instrument carefully, and with the bubble central take staff readings a and b on the same staff held respectively on the pegs A and B. The true difference of level will be $x = a \sim b$.

(2) Set up the level firmly at C, levelling it carefully; and with the bubble central read the staff held on A and B, thus obtaining readings a' and b', so that the false difference of level is $y = a' \sim b'$ (see Note). If $x = y$, the line of collimation is parallel to the bubble line.

If the bubble traverses fairly well, proceed with the adjustment of the diaphragm, if necessary; but if the collimation is correct ($x = y$) the level tube should not be disturbed unless a traversing bubble is essential, since interference with the level tube will vitiate the collimation adjustment.

(3) Assuming that the collimation adjustment is in error, first correct the bubble by means of the level tube screws, making the half and half adjustment over the one tribrach screw (or a pair of plate screws); turn the telescope through 90° and centralise the bubble solely by the two tribrach screws parallel to the telescope (or plate screws beneath the telescope). Repeat if necessary until the bubble traverses.

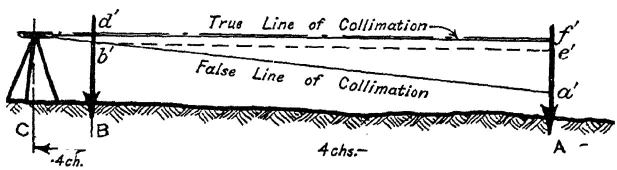


FIG. 10.

(4) Again take the readings a' and b' and find the false difference of level $y = a' \sim b'$. The error in $AB = 4$ chains is $E = x \sim y$, while the error to be corrected is in 4·40 chains, and is therefore $11/10\ E$, which is $a'f'$ in Fig. 10. Ascertain the reading f', and, sighting a target (improvised or otherwise), raise (or lower) the diaphragm until the horizontal hair comes to the reading f', slackening one screw and taking up the slackness with the other. Since the image is inverted, the diaphragm must be lowered for a higher reading and raised for a lower reading.

(5) Finally, sight the staff held on B, obtaining the reading d', and if $f' \sim d' = x$, the adjustment is in order. Otherwise repeat the unbalanced sights, and adjust accordingly.

Note. Some surveyors set up the level over the peg B, so that the reading b' is a staff measurement up to the eyepiece. This practice is followed when reciprocal sights are taken in the contingency of no chain or tape being at hand.

(B) **Old treble adjustment level.** (1) Carry out the two-peg test, as described in (1) and (2) preceding.

If the collimation adjustment proves to be in order, the bubble may be made to traverse, if necessary, by the conjoint use of the limb screws and plate screws. If, however, the attachment of the telescope to the limb is not perfectly stable (as might well be the case with an old level) it may be advisable to secure this, making the instrument effectively a double adjustment level, which is adjusted in the manner heretofore described. If, however, the collimation adjustment is in error, but the bubble traverses, the necessary correction may be made with the diaphragm screws likewise. If both collimation and bubble are at fault, it is advisable to place the horizontal line in the centre of the field of view, provided the limb attachment is rigid, as follows :

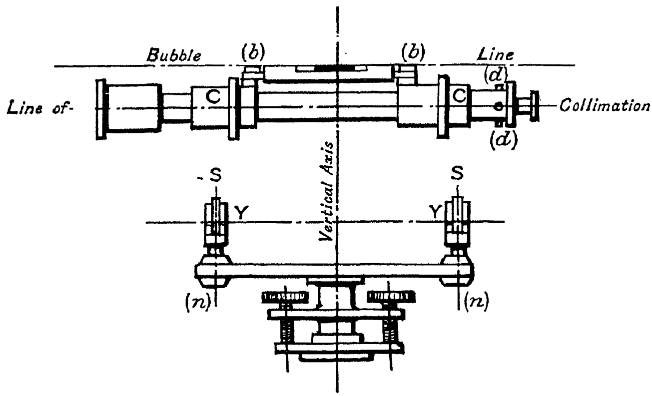
(2) Sight the staff at a distance of about 30 ft., focus the image and pass the eye round, reading the maximum vertical limits of the field of view. Ascertain the mean reading, and bring the horizontal hair to this reading by means of the diaphragm screws.

(3) Now make the bubble traverse by conjoint use of the plate and limb screws. Observe the readings a' and b' of the unbalanced sights from C, ascertaining the false difference of level and thence the reading f' on the staff A corresponding to $\frac{11}{10}(x \sim y)$. Bring the horizontal hair to the reading f' by means of the limb screws, and take the reading d' on the nearer staff B. If $f' \sim d' = x$, the line of collimation is truly horizontal. Otherwise repeat the test and adjustment.

(4) Finally, make the bubble traverse by conjoint use of one pair of plate screws and the level tube screws ; turn through 90° and centralise the bubble with the other pair of plate screws alone, repeating if necessary.

Note. Some writers still suggest Gravatt's three-peg test, though imperfections in the draw tube are not always amenable to correction in this way. Further, it can be shown that it is not an essential condition of accurate levelling that the intersection of the cross-hairs should lie exactly in the axis of the telescope provided the travel of the focussing tube is straight.

(C) **Single adjustment levels.** (1) Carry out the two-peg test, as described in (1), (2) with reference to the double adjustment type, levelling up at L and C to the circular bubble by means of the tribrach screws, one of which should desirably be under the telescope when sighting along AB. (If a micrometer screw is provided, set this at zero.) (2) Find the collimation error $E = x \sim y$ as described, and determine the reading f' on the distant staff A corresponding to $\frac{11}{10}E$. (3) Set the horizontal line of the telescope to the reading f', using the foot screw under the telescope, and the tilting screw for the final setting, if necessary. Adjust the main bubble to its central position. Check the reading f' and the reading d' on the peg B.



. 11. Y Level.

(If the micrometer does not now read zero, slacken the holding screw and rotate the head to zero position. If the micrometer is merely used as a quickset screw, this correction is unnecessary, though generally no great deviation from zero should accrue.)

*Example**. Detail concisely how you would test and, if necessary, make the permanent adjustments of a Y level.

Collimating. (1) Set up the instrument about 250 ft. from a wall or building, and level it up approximately with the foot screws, keeping one tribrach screw (or a pair of plate screws) underneath the telescope in the proposed direction of sighting. (2) Carefully focus the cross-hairs, and, sighting the wall, direct the marking of a cross (+) as a test point P. Open the straps S, and rotate the telescope in the Y's. If the centre of the horizontal hair does not remain fixed on the image of the point P, the line of collimation is not coincident with the axis of the telescope. (See Note *a*, p. 21.) (3) Assuming that this is the case, return to the normal position, and (if necessary) bring the centre of the cross-hair to the point P. Rotate the telescope half round in the Y's and direct the marking of a point Q corresponding to the centre of the cross-hair. Measure the vertical deviation PQ on the wall, and bisect PQ at O, OP being the collimation error in the sight length. (4) Bring the centre of the cross-hair on to the mark O by loosening one diaphragm screw (d) and taking up the slackness with the other. Verify the adjustment.

End for end reversals. (5) Level up the instrument, keeping the telescope over one tribrach screw (or pair of plate screws). Centralise the bubble with this screw (or pair). Lift the telescope bodily from the Y's and replace it in them end for end. If the bubble remain central the bubble line is parallel to the bearing surfaces of the collars C, which by construction should be equal cylinders coaxial with the axis of the tele-

scope. Otherwise, correct half the bubble error with the level tube screws (*b*) and the other half with the tribrach screw (or plate screws) underneath the telescope. Verify the adjustment, and repeat if necessary. (See Note (*b*).) (6) Replace the telescope normally in the Y's and secure it with the straps.

Traversing. (7) Set the bubble central with the tribrach screw (or plate screws) underneath the telescope. Turn through 90°, and, if necessary, set the bubble central with the foot screws now parallel to (or underneath) the telescope. Return the telescope back through 90°, and, if necessary, reset the bubble with the tribrach screw (or plate screws). (8) Turn the telescope end for end over this one tribrach screw (or pair of plate screws), and make the half and half adjustment of the bubble by means of this and the Y nuts (*n*).

Turn the telescope through 90°, and set the bubble central solely by means of the two foot screws parallel to (or underneath) the telescope. Repeat this *plate reversal* test and adjustment if the bubble does not now traverse.

Notes. The construction of the modern reversible level can be appreciated only with an intimate knowledge of the Y level ; though, unfortunately much of the technique is necessarily omitted from the foregoing procedure.

(*a*) Readings on a vertical staff may be taken in place of the marks *P*, *Q*, and *O*, and in this connection it may be noted that the line of collimation may be correct in the vertical plane, but not in the axis of the telescope, as would be evident upon rotation on the image of a nail, or pin-point of light. The fault is due to incorrect centering of the diaphragm, but is not of serious importance.

(*b*) If azimuth screws are also fitted to the level tube, it may be necessary, as a preliminary step, to place the bubble line in the plane of the axis of the telescope by means of these. Unequal collars will not affect the end for end reversal test, but the maladjustment would be evident in the two-peg test.

Example.* Describe how you would test and, if necessary, adjust one of the following original types of reversible levels: (*a*) Cooke's, (*b*) Cushing's.

Normally in these instruments the line of collimation is set perpendicular to the vertical axis by *telescope reversals*, and the bubble line is then made perpendicular to the vertical axis, the latter being set truly vertical in the final process of plate reversals.

(*a*) Cooke level.

Collimating. With the stop screw withdrawn, collimate in the manner described for the Y level (p. 20) with the instrument levelled approximately. (Utilise Fig. 11.)

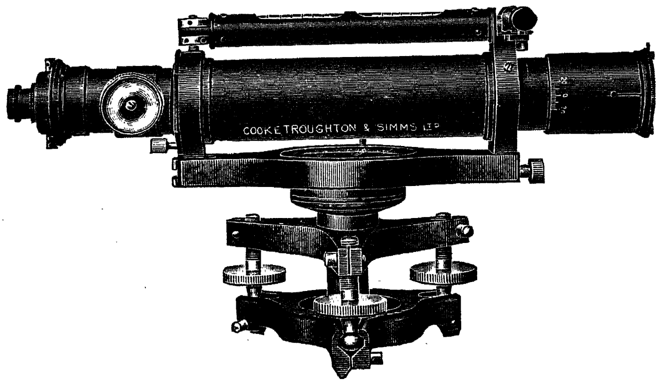


FIG. 12. Cooke's original Reversible Level.

Telescope reversals. (1) Sight a staff held at a distance of 200–250 ft. from the instrument, which should be levelled up approximately with a tribrach screw underneath the telescope. Take a staff reading *a*. (2) Withdraw the telescope from its socket *S*, turn the socket about the vertical axis so that it lies end for end in the same direction. Insert the telescope so that the object glass is towards the staff. With the socket *S* thus reversed on the telescope, note that the cross-hair is horizontal with the focussing screw on the same side as when the reading *a* was taken. Take a staff reading *b*. If the readings *a* and *b* are identical, the line of collimation is perpendicular to the vertical axis. (3) Assuming that this is not the case, find *o*, the mean staff reading. Slacken the base plate nuts (*nn*) and raise or lower the socket at that end until the cross-hair comes to the reading *o*. (4) Withdraw the telescope and insert it in its normal position in the socket, and if the same staff reading *o* is obtained, finally tighten the nuts (*nn*) in such a way that no disturbance results. Verify the test and adjustment if necessary. Insert the screw.

Traversing. (5) Set the bubble central with the tribach screw underneath the telescope. Turn through 90° and, if necessary, centralise the bubble with the two screws now parallel to the telescope. Turn the telescope back through 90°, and, if necessary, reset the bubble with the tribrach screw underneath the telescope. (6) Turn the telescope end for end over this one tribrach screw, and make the half and half correction of the bubble with this screw and the level tube nuts (*bb*). Turn the telescope through 90°, and centralise the bubble solely by means of the two screws parallel to the telescope. Repeat the test and adjustment if the bubble does not now traverse.

(*b*) **Cushing level.**

The processes of collimating and making the bubble traverse are identical with those of the foregoing level.

Telescope reversals. (1) Take the staff reading a as above. (2) Reverse the telescope by interchanging the objective and eyepiece ends; take the reading b, and find the mean reading o. (3) Proceed as in the corresponding step for the Cooke Level, reading " limb " nuts for " base-plate " nuts. (4) Interchange the objective and eyepiece ends, attaching them so that the telescope is again normal. Repeat the test and adjustment if the same staff reading o is not obtained.

Note. Instead of staff readings, test marks a, b, and o may be made vertically on a wall, a similar mark having been used in collimating.

Example. Detail the method of adjusting a Zeiss pattern level, explaining clearly how the errors are eliminated in the various steps of the procedure.

Zeiss pattern level. The fundamental lines of a modern reversible level are: (*a*) The bubble axis, or virtual line of the two lines of the reversion level, and (*b*) the line of collimation. (i) The construction must ensure that the line of collimation shall be in the coincident mechanical and optical axes of the telescope. (ii) The bubble axis must be parallel to the line of collimation.

(1) Plant the tripod with the head fairly horizontal, preferably with a foot screw under the telescope when sighting the staff. Using the foot screws, centralise the circular bubble. (2) Sight a staff at a distance of (say) 150 ft. or 3 chains; and, using the tilting screw, bring the bubble ends into coincidence, as viewed in the sighting prism. Record this reading for the *normal position* (I). (3) Rotate the telescope in its sleeve through 180° to the *inverted position* (II), and reset the mirror. Bring the bubble ends into coincidence by means of the tilting screw, and note the staff reading. (4) Attach the cap to the normal objective, and remove the eyepiece to this cap, focussing it correctly. Rotate the telescope about the vertical axis to the *reversed position* (III), and again read the staff with the bubble ends coincident. (5) Rotate the telescope in its sleeve to the *subnormal position* (IV); and finally read the staff with the bubble ends set coincident by means of the tilting screw. (6) Remove the cap, and insert the eyepiece in its normal position; then, by means of the tilting screw, set the cross-line to read on the staff the mean value of the readings in I, II, III, and IV. Now that the line of collimation is truly horizontal, slacken the locknut and, turning the prism-box screw, bring the bubble ends into coincidence. The instrument will then be adjusted for use in its *normal position.*

If, however, the required movement is beyond the range of motion of the prism box, adjust the bubble between the end marks by means of the vertical level tube screws, taking care that the prism box is in its mean position. Make the final adjustment with the prism-box screw, leaving

the locknut firm. (7) Repeat the test in the four positions, particularly whenever the level tube has been adjusted.

Theory. On the assumptions that coincident images correspond with a horizontal bubble line, and that the angle between the upper (1) and the lower (2) bubble lines is α, upwards, or plus, in the direction of sighting, then for the four positions :

I. Let the line of collimation be inclined upwards at θ_1, giving an error $+\theta_1$.

II. On rotating the telescope through 180° and setting the now uppermost bubble line (2) horizontal, the collimation error will be $+(\alpha-\theta_1)$.

III. On reversing the eyepiece, the angle θ_1 may be altered to θ_2, and on setting the bubble line (2) horizontal, the collimation will be inclined downwards with an error of $-(\alpha-\theta_2)$.

IV. Finally, on rotating the telescope, and setting the upper bubble line (1) horizontal, the line of collimation will be inclined downwards with an error $-\theta_2$.

Wherefore the average error for the four positions will be zero, and the corresponding average reading on the staff will accord with a horizontal line.

If the level is in adjustment, it is evident that the reading I should be the mean of the readings II, III, and IV.

In ordinary work all readings are taken with the telescope in the normal position, though in more precise work the mean of I and II is frequently recorded, and, in this case, the prism box is so adjusted that the mean of the readings in positions I and II agrees with the mean in the four positions.

If the axis of the level tube is not parallel in plan with the axis of the telescope, a slight twist of the latter will cause a displacement of the bubble. Adjustment by trial and error is effected by means of the horizontal pair of level tube screws.

Example. Zeiss pattern level, L_3. Indoors, 75 ft. distant.
Observed : I...5·662 ; II...5·667 ; III...5·641 ; IV...5·686. Mean : 5·664.
Checked : IV ... 5·686 ; III ... 5·643 ; II ... 5·668; I ... 5·663. Mean : 5·665.
Set cross-line at 5·665 on staff and adjusted bubble with prism screw.
Verified : I...5·665 ; II...5·660 ; III ... 5·648 ; IV...5·665. Mean : 5·6645.

Example. Describe how you would test and adjust the following modern reversible levels : (*a*) Cooke's improved ; (*b*) Watts' self-adjusting.

(*a*) **Cooke's improved level.**

This instrument, a development of the Improved Engineers' Level, introduces a reversion level of so accurate a type that, in conjunction with precision construction, only the normal and inverted positions of a normal telescope, I and II (p. 11), are concerned in adjustment tests. The bubble

is viewed from the eye end of the telescope in a compound speculum mirror.

The fundamental lines are those of a modern reversible level, as also is the condition of adjustment open to the surveyor, the mechanical and optical axes being constructed with perfect coincidence. The means of adjustment are the level tube screws at the normal objective end of the level tube.

Method of adjustment. To make the bubble line parallel to the line of collimation. (1), (2) These are similar to the steps prescribed for the Zeiss level, though considerable latitude is also permissible in the sight length. With the telescope normal, set the micrometer screw to read zero after levelling up approximately to the circular bubble. Bring the bubble to its mid-position by means of the micrometer screw, and read the staff (I). (3) Invert the telescope, and bring the bubble to its mid-position by means of the micrometer screw. Read the staff (II). (4) Rotate the telescope to its normal position (I), and by means of the micrometer screw bring the horizontal line of the diaphragm to read the mean of I and II on the staff. Adjust the bubble to its mid-position by means of the vertical level tube screws. (5) If it is desired to use the micrometer as a gradiometer, set it to read zero, if necessary by slackening the holding screw and rotating the divided portion to zero reading.

(*b*) **Watts' self-adjusting level.**

Characteristic of Watts' self-adjusting levels is the constant bubble of the reversion type (p. 9) and the simple prism device, which is duplicated for observing one end only of the normal and inverted bubble. The instrument is made in the 11″ and 14″ and 21″ patterns, all of which are provided with a micrometer tilting screw, approximate levelling by the tribrach being controlled by a circular bubble.

The fundamental lines and condition of adjustment are those of the modern reversible level, and the means of adjustment is a screw at the normal objective end of the level tube, a differential screw being provided in the largest model, where the circular bubble is superseded by cross-levels.

The method of adjustment is essentially the same as that described for the foregoing instrument, except that only one end of the bubble is viewed.

QUESTIONS ON ARTICLE 3

1*. A Zeiss level is believed to be out of adjustment, and it is found that the lens cap is missing, so that the eyepiece cannot be inserted at the reverse end of the telescope.

Describe in detail how you would verify the adjustment under the circumstances. (U.L.)

[Resort to two-peg test, and if necessary adjust as a single adjustment level by correcting the bubble.]

2*. On completing a circuit of levels you are convinced that your dumpy level is out of adjustment.

Describe how you would verify this, giving fictitious staff readings, when no chain or tape is available and there is no stretch of still water in the neighbourhood. (U.L.)

[Take reciprocal levels, measuring up to the eyepiece.]

3*. Comment upon the following statements, giving appropriate diagrams :

(*a*) " The only absolute test of the accuracy of adjustment of a level is fundamentally the two-peg method."

(*b*) " It is not an essential condition of accurate levelling that the cross-hair should lie in the axis of the telescope, provided the hair moves in a straight line in focussing." (U.L.)

[*Hint.* Draw an object glass with axis and cross-hairs out of axis and at small angle thereto. Assume points X, Y, on the line of travel of the cross-hairs, draw rays from X, Y, parallel to the axis to converge through the principal anterior focus, and insert the unrefracted rays from X, Y to meet the former rays in points E, H. The line EH will be straight if the travel XY is straight.]

4*. As engineer in charge of a public works contract, you wish to establish marks on a concrete surface so as to test the accuracy of your dumpy level from time to time.

Figure on a sketch fictitious readings for these marks, and show the staff readings before and after adjusting your level. State also any other adjustments you would consider before or after the above test (*a*) when only the bubble is also adjustable ; and (*b*) when additional means of adjustment are provided beneath the telescope. (U.L.)

5†. In levelling up a hill-side, the sight lengths were observed with the stadia lines, the average length of the *ten* backsights and *ten* foresights being 70 ft. and 35 ft. respectively.

Since the observed difference of reduced level of 78·40 was disputed, the level was set up midway between two pegs, A and B, 300 ft. apart, and the reading on A was 4·60 and on B 5·11 ; and when set up in the line AB, 30 ft. behind B, the reading on A was 5·17 and on B 5·64. Calculate the true difference of reduced level. (U.L.)

[Collimation slopes upwards, $+0{\cdot}04$ in 300 ft., and total error is $0{\cdot}000133(70-35)\times 10 = 0{\cdot}047$ subtractive.]

6†. The diaphragm of a dumpy level was lowered to its limit, bringing the line of sight considerably out of the axis of the telescope. The line of sight thus displaced was then set truly horizontal and the bubble was brought to the middle of its run ; and the instrument so adjusted gave satisfactory results on a number of test circuits.

Can you explain this in view of the adverse object of Gravatt's " three-peg " test, and, further, elucidate the contention that this test will facilitate adjustment so as to eliminate the combined effects of curvature and refraction? (U.L.)

[There was no droop or other defect in the draw tube, and in consequence the adjustment holds, even though the cross-hair is not in the axis of the telescope.

Gravatt's test involved balanced sights between two pairs of pegs, AB and BC, in the same straight line, with $AB = BC$, the respective readings being a, b, and c. Unbalanced sights were then taken from D, in AC, say $\frac{1}{2}$ chain behind C, the corresponding readings being a', b' and c'. Then if

$$(c' - c) - (a' - a) = 2(b' - b) - (a' - a),$$

the level is in adjustment for collimation. Otherwise the diaphragm was adjusted until such was the case (Heather's "Mathematical Instruments" 1851). It was also contended that when the bubble line was set parallel to the line of collimation the instrument would be in "complete practical adjustment" also for curvature and horizontal refraction for distances up to ten chains, the maximum error being less than 1/1000 ft.

It will be found normally that the error of sighting exceeds the effects of curvature and refraction, which incidentally vary as the square of the sight-length.]

7*. A re-conditioned Y level was found to be inaccurate, although responding to the usual tests for this type of level. On examining the collars, however, it was found that the one nearer the eyepiece was 0·0028″ smaller than the other, the centres of the collars being exactly $7\frac{1}{2}$″ apart. The level tube was attached to the telescope with capstan nuts.

State clearly (a) how you would use the instrument without adjustment, and (b) how you would adjust the instrument so that it might be used in the ordinary way, in each case confining your answer to the adjustment in error. (U.L.)

8*. Discuss with reference to sketches the essential differences in the procedure of effecting the permanent adjustments of the Y level and Cooke's original reversible level.

9*. Draw a neat longitudinal section of the telescope of a Zeiss level, indicating the various parts ; and state concisely how you would make the permanent adjustment of this level. (U.L.)

10*. The following readings were taken with a Zeiss level on a staff approximately 50 ft. from the instrument, the roman numerals indicating the four positions of the telescope :

I, 5·275 ; II, 5·295 ; III, 5·235 ; IV, 5·235.

State clearly how the adjustment was made if necessary.

The two-peg test was then made as a check on a line AB, (a) equal sights of 2 chains being taken first ; and then (b) sights from a point C, 40 links behind B in the line AB.

(a) On peg A, 5·02.	(b) On peg A, 4·29.
,, B, 5·04.	,, B, 4·29.

State your opinion as to the accuracy of the adjustment. (U.L.)

[Adjustment to mean reading of 5·260 gives a collimation error of 0·022 feet in 4·40 chains.]

11†. The notes relative to the test and adjustment of a Zeiss reversible level were recorded as follows :

I	II	III	IV
4·306	4·326	4·274	4·292
4·300	4·312	4·288	4·298

Explain these, describing the tests to which they apply. Also state if the level was adjusted adequately for engineering surveys. (U.L.)

12. Describe fully, with sketches, the adjustments of the dumpy level. (U.D.)

13. Describe the procedure in adjusting a wye level. (U.D.)

14. To test the line of collimation of a dumpy level the instrument was set up equidistant from two pegs A and B, which are 200 feet apart. The respective readings on a staff held on A and B are 3·71 feet and 5·82 feet. The instrument was then moved to a station C, 50 feet from A and 250 feet from B. The staff reading on A was now 5·86 feet and on B was 7·35 feet. Calculate the reading on staff A to which the line of collimation should be set to make the line of sight horizontal. (I.C.E.)

[6·015.]

15. A wye level is set up firmly, and the telescope is directed so that the intersection of the cross-hairs appears to lie upon a small mark. The telescope is then rotated in its supports through 180°—so that the level tube is brought from the upper to the lower side of the telescope axis—and the intersection of the hairs is seen to leave the mark. Explain the nature of the error which is revealed, discuss its importance, and describe how you would eliminate it by adjustment. (I.C.E.)

16. Explain why the lengths of sight should be kept approximately equal while carrying out levelling where great accuracy is required.

Describe fully the adjustments of a dumpy level. (I.C.E.)

17. Explain, with sketches, where necessary, what is meant by the following in levelling operations :

(*a*) Fundamental Bench Mark,
(*b*) Ordnance Datum,
(*c*) Reciprocal Levelling.

The following observations were taken during the testing of a dumpy level :

(*a*) Instrument equidistant from pegs A and B.
Staff readings on A and B : 4·16 and 7·14 ft. respectively.

(*b*) Instrument on the line BA produced and 100 distant from peg A.
Staff readings on A and B : 3·84 and 8·00 respectively.
Distance between pegs A and B = 384 ft.

Calculate the staff readings which should be obtained in case (*b*) to give a horizontal line of sight, and describe briefly how the instrumental adjustments can be completed. (U.B.)

[On A, 3·53(3) ; on B, 6·51(3).]

18. Describe in detail the adjustments that have to be carried out to put a gradienter level in good working order.

Also describe carefully the routine that has to be followed and the precautions to be taken in running a line of levels.

Reasons must be given for all statements. (U.C.T.)

ARTICLE 4 : THEODOLITES

Theodolites used to be distinguished as being either *plain* or *transit* theodolites, the former being the original pattern of small instruments, " Y " theodolites as introduced by Jonathan Sissons. Although representative instruments, such as the plain and Everest theodolites, are still obtainable, the tendency is to introduce the transit principle on the smallest instruments.

Present-day instruments may be placed in the following categories, the size being normally determined by the diameter of the reading edge of the horizontal circle, as given in brackets.

(1) **Geodetic theodolites,** (8″ to 15″), invariably with micrometer microscopes or special optical devices for reading the circles. Small instruments are now superseding these larger models.

(2) **General survey theodolites,** (4″ to 6″), including a wide range of engineering and survey models, such as the (*a*) Tavistock and Wild, (*b*) general purpose, (*c*) Colonial patterns, and (*d*) tunnelling, mining, and railway theodolites.

(3) **Light theodolites,** (3″ to 4″), frequently styled geographers', prospectors' and builders' theodolites.

Colonial theodolites are often vernier instruments, and railway theodolites are not provided with vertical circles, the pattern being styled the plain transit in America. Tacheometers are theodolites with specific features for facilitating tacheometrical observations. Instruments which utilise the " tangent " principle are readily classified, but those introducing the stadia, or subtense, principle may often be styled theodolites, although originally the addition of an anallatic telescope was the characteristic feature.

The theodolite consists of three primary portions, subdivided into components, as follows :

The alidade. (i) The **vertical,** comprising fundamentally the telescope and transverse (or horizontal or trunnion) axis which rests in the bearing of (ii) the **standards,** these latter rising vertically from (iii) the **upper plate,** the inner spindle of which rotates in the hollow spindle of the *limb*.

In complete instruments, the vertical includes the vertical circle, the altitude (or azimuthal) level, and the clipping frame, which carries the verniers or microscopes of the vertical circle together with the clamping screw and slow-motion screw, hereafter, for brevity, styled the *vertical motion*. The circle is usually graduated in the *quadrant division* (0°–90°–0°–90°–0°) in engineers' instruments, though the tendency (often without reason) is to introduce the *whole* or *half*-circle division, the zero reading occurring when the zenith is sighted. The telescope, which is either tube or internal focussing, is usually fitted with a diaphragm webbed with a vertical and a horizontal line, a subtense interval being often provided. The upper plate carries the verniers or microscopes for reading the horizontal circle, also the plate levels for the ordinary levelling up of the instrument. Details essential to adjustments, etc., are (*a*) the clipping screws with which apparent index error of the vertical circle is eliminated, (*b*) the standard nuts, with which the transverse axis is adjusted, (*c*) the altitude level adjusting nuts, (*d*) the diaphragm screws, and (*e*) the plate level screws for the relevant adjustment.

The upper plate is locked to the limb and moved relatively thereto by means of a clamp and tangent screw, hereafter called the *upper motion*.

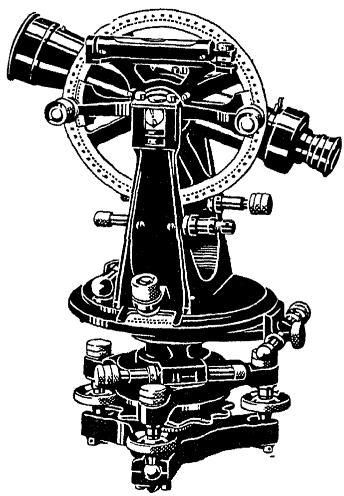


FIG. 13. Stanley GS2 Theodolite.

The limb. This consists substantially of the horizontal circle, usually *whole* circle division (0° to 360° clockwise), and, perpendicular to this, the outer hollow spindle which, enclosing the inner spindle, rotates in the levelling head.

The limb is locked to the levelling head and moved relatively thereto by means of a clamp and tangent screw, hereafter called the *lower motion*.

The levelling head. This component serves the three-fold purpose of providing a bearing for the outer hollow spindle, the means of levelling the instrument, and the means of attaching the instrument to its tripod. It may also embody the shifting stage, or centering arrangement, with which the suspended plumb bob is centred over a point.

Double-reading microscopes. The outstanding feature of modern theodolites is the introduction of prisms which bring the images of opposite parts of the circles into the field of view of the same microscope, thus avoiding the disturbances that arise from walking around the instrument. The principle is represented diagrammatically in Fig. 14,

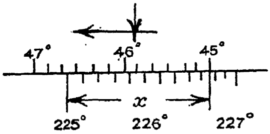


FIG. 14 (*a*).

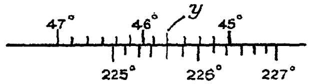


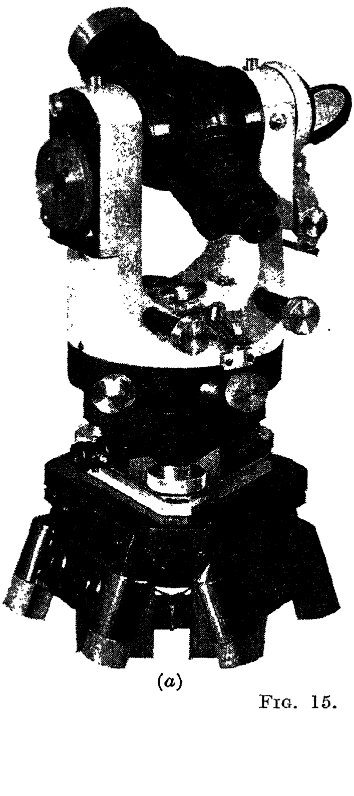
FIG. 14 (*b*).

where views of opposite parts of the circle are shown, the index being indicated thus : ↓ .

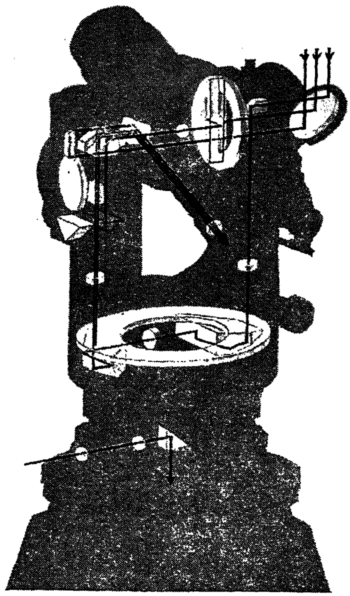
Since the images are read at the same index, the average reading is $\alpha° + \frac{1}{2}x$, where x is the distance between the corresponding divisions ($\alpha°$ and $180° + \alpha°$), which have not over-run each other.

Thus, in Fig. 14 (*a*), x is 9·6 divisions of 10′ and the average reading is $45° + \frac{1}{2}(9{\cdot}6 \times 10') = 45°\ 48'$. In order to read to greater precision, some type of estimating microscope is used, as in the instruments referred to below.

Zeiss and Wild theodolite. In the Universal theodolite the circles are graduated on glass cylinders, and in consequence, smaller diameters, finer



(*a*)



(*b*)

FIG. 15. Zeiss Theodolite.

division, and higher magnification may be used. The microscope eyepiece is beside that of the telescope, and by interposing a prism, the vertical circle readings are also made with the same microscope. Effectively the two images of the circle are made to coincide by moving them by equal amounts in opposite directions. This is done by passing the rays from opposite portions of the circle through two parallel-plate micrometers which rotate by equal amounts in opposite directions, the motion of each being read on a drum divided in seconds from 0′ to 10′ in the same microscope. Fig. 14 (*b*) shows the image of the circle when brought into coincidence from the position of (*a*), the motion y, as indicated, being the distance between the corresponding divisions which have not over-run (45° 40′ and 225° 40′). The average reading is 45° 40′ (as suggested by the position of the index) *plus* the increment $\frac{1}{2}y$ obtained from the micrometer in minutes and seconds, say, 12′ 22″.

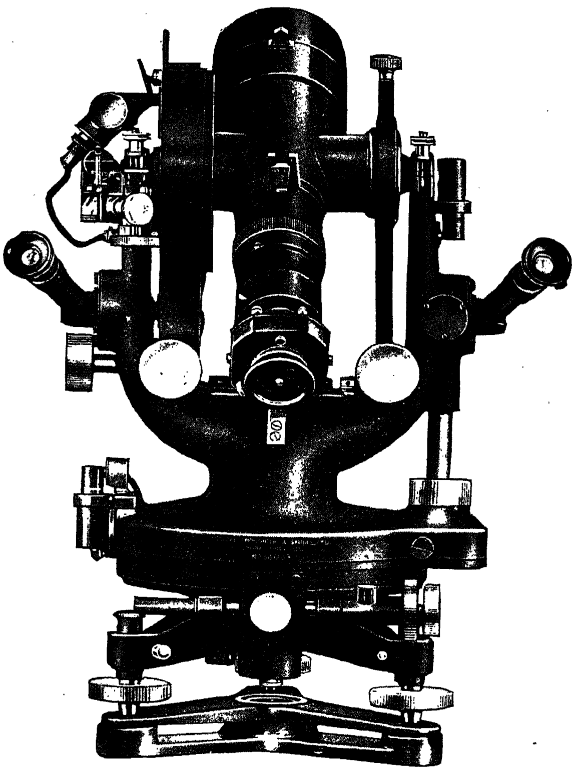


Fig. 16. "Tavistock Theodolite."
(Messrs. Cooke, Troughton & Simms.)

Casella's theodolite. In this instrument certain features of the well-known micrometer microscope are retained; namely, the smallest divisions on the circle are 10′ and the smallest on the drum 10″, a complete rotation of the drum thus corresponding to 10′. But to represent the opposite portions of the circle, the drum has numbers complementary to its 1′ divisions for the two readings in opposite directions, thus:

6 5 4 3

4 5 6 7

Hence when the micrometer is brought to the nearest lower reading, say 45° 40′, the micrometer reading might be 9′ 20″, and when brought to the corresponding 225° 40′ division, it might be 9′ 30″, giving an average reading of 45° 40′ 25″. The micrometer drum appears in the same microscope as the horizontal circle reading, an adjacent microscope showing likewise the vertical circle.

The "Tavistock" theodolite. An entirely different form of micrometer is used for double reading to single seconds in this well-known instrument of Messrs. Cooke, Troughton, and Simms. Two microscopes are fitted, one on each side of the telescope for separate reading of the horizontal and vertical circles, the construction allowing the telescope to be transitted.

Each microscope shows a screen with three apertures, as follows: (1) Reading to the next lower 1° and 20′; (2) additional fine reading in minutes and seconds; and (3) index mark, finally bisecting the interval between corresponding opposite graduations.

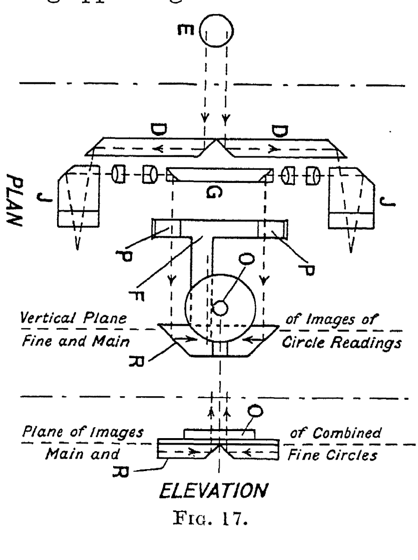
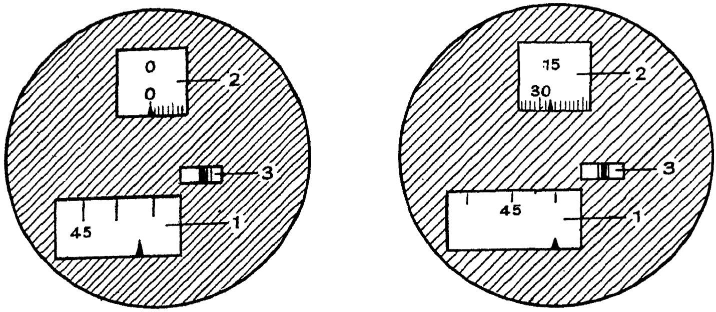


Fig. 17.



(a) FIG. 18.

The horizontal and vertical circles are respectively $3\frac{1}{2}''$ and $2\frac{3}{4}''$ in diameter; both are divided on glass to 20′, are silvered, and are read through the glass, the construction protecting the silver from tarnishing. A head is provided outside, and, through the medium of a rack and pinion, a frame rotates a glass fine-reading circle O in the plane of the images of the circle divisions, the reading circle being divided in seconds from 0′ 0″ to 20′ 0″. This frame also operates travelling prisms (P) which with prisms (R) comprise the essential part of the optical device (Fig. 17). The main index mark of (3) is at the junction of the prisms R, while the line of separation of the images of opposite parts of the circle is parallel to the graduations, which increase in the same direction.

The images of the graduations on opposite sides of the circle are brought together by reflection through the prism system J, D, and R, with the line of separation of the images parallel to the graduations. By means of the travelling prism P, the images of the main and fine reading circles can be moved by equal amounts in the same direction by movements of the frame F, the movement of the images being proportional to the movements of the travelling prisms. The screen with its three apertures is shown in Fig. 18 (*a*) and (*b*) as it would appear through the microscope, the numbers at the apertures being as already explained.

Fig. 18 (*a*) shows the screen after the observation of an angle. The index in (1) suggests that the angle is greater than 45° 20′, (2) reads 0′ 0″, while in (3) only one graduation appears with the main index mark. If now the micrometer head is turned, moving both images to the right until two graduations appear equally spaced on each side of the index mark in (3), the nearest 20′ division appearing at the coarse index in (1), then the fine reading will be given in (2), say 15′ 32″, giving a complete reading of 45° 35′ 32″.

Concisely, the process consists in observing the next lower angle to the 20′ division in (1), turning the head until opposite divisions are equidistant from the main index in (3), and reading the values in apertures (1) and (2) for the angle to the nearest second.

Single-reading microscopes. The micrometer microscopes are fitted in holders in the positions otherwise occupied by the verniers ; that is, in pairs, 180° apart, those for the horizontal circle being attached to the upper plate, and those for the vertical circle to the clipping frame. Each microscope consists of a small microscope penetrating a flat rectangular body, which encloses (1) the *micrometer screw*, (2) the *diaphragm*, and (3) the notched *index plate*, a micrometer head outside actuating the screw and so moving the diaphragm at right angles to the axis of the microscope tube. The microscope is fitted with an adjustable eyepiece, also an open sleeve which may be rotated so as to admit light to the circle. A milled head is provided for turning the micrometer screw, the divided head being usually a friction fit.

The field of view of the microscope exhibits a portion of the graduated circle, which is exceedingly finely divided, each degree being figured in ordinary instruments and divided into six parts. Over this is seen the notch of the fixed index plate, and, across the circle, the twin vertical lines of the glass diaphragm, the lines moving in response to the micrometer head, which is often divided into sixty parts, the ten primary divisions being figured. If the microscope is in perfect adjustment, (*a*) the micrometer head will read zero when the vertical lines exactly enclose a division, (*b*) the V notches will be in the centre of the field of view and exactly 180° apart, while (*c*) the microscopes will be so adjusted in their holders that the magnification is correct and one turn of the head moves the vertical lines exactly one division of the circle. Ordinarily the microscopes are open to three adjustments corresponding to (*a*), (*b*), and (*c*) above, the third being essential to accuracy and the others convenient and desirable.

Now if f is the focal length of the objective of the microscope, f_1 the distance from that lens to the circle, and f_2 the distance between the objective and the image of the circle within the tube ; then when $f_1 > f < 2f$, $f_2 > f_1$, and magnification results in the ratio f_2/f_1 or $f/(f_2+f)$ diameters, since $1/f_1 - 1/f_2 = 1/f$. Thus the image of a fixed dimension, a circle division, is magnified in the ratio f_2/f_1, where f_2 varies with f_1, or $f_2 = f_1 f/(f - f_1)$; and if the image is too large or too small, f_1 must be increased or decreased respectively, to an extent not readily commensurate but easily found by trial.

In general, the lowest reading x is found as follows : *Ascertain c the angular value of the division, or divisions, of the circle traversed by the lines on rotating the head one complete turn, and divide c by the number of divisions n on the micrometer head. Thus* $x = c/n$. Hence if the circle is

divided into 360×6 parts finally, and the head likewise into 60 parts, the smallest direct reading on the circle will be 10′, corresponding to one rotation of the head, and the least direct reading with the micrometer will be 1/60 of this, or 10″, though obviously a lower reading is obtainable by interpolation—a doubtful refinement in smaller instruments.

The process of reading an angle is concisely as follows: read directly the next lower circle division from the notch, bring the vertical lines back to this division, and add the micrometer reading to the direct reading.

QUESTIONS ON ARTICLE 4

1†. Describe with neat sketches one of the following devices for reading submultiples of the smallest divisions of divided circles: (*a*) micrometer microscopes; (*b*) Zeiss or Wild methods.

State the merits and defects of the device in particular regard to engineering setting-out work. (U.L.)

2†. Explain with the aid of neat sketches the optical arrangement for the precise reading of the horizontal circle of *one* of the following instruments: (*a*) Zeiss theodolite; (*b*) Tavistock theodolite. (U.L.)

3. Draw a vertical section through a Zeiss theodolite, indicating the path of the rays from the horizontal circle, and explain with sketch how the average reading of the horizontal circle is obtained to 1″.

4. Explain what recent developments have taken place in the construction of a modern type of theodolite. What are the advantages over the older type of construction? (I.C.E.)

5. Describe clearly with *neat* sketches a micrometer microscope as used for reading the horizontal and vertical circles of a theodolite.

What degree of accuracy of reading is usually given by such an instrument? (I.C.E.)

6. Describe with the aid of sketches the optical system used for reading the horizontal circle of either the Wild or Tavistock theodolites. State clearly the function of each lens, prism or other optical device used.

Also describe in detail the adjustment for the elimination of index error in the instrument selected. (U.C.T.)

7. Compare the construction of a vernier theodolite with one of the modern 20″ theodolites such as the Zeiss, Wild, and Cooke's Model T63, and discuss their advantages and disadvantages.

The diaphragm of a theodolite in good adjustment is broken and replaced by the surveyor.

What tests and adjustments must he carry out in order to bring his instrument in good working order again? (U.C.T.)

8. (*a*) Describe with the aid of sketches two devices for reading the horizontal circle of a theodolite, and discuss their advantages and disadvantages.

(b) Explain how it is possible to set up a theodolite truly level if all bubbles are out of adjustment and no equipment is available for adjusting the theodolite.

(c) State, with reasons, what error in the adjustment of a theodolite is most troublesome when the instrument is used as a tacheometer, and describe briefly how the adjustment of the error may be carried out. (U.C.T.)

ARTICLE 5: ADJUSTMENT OF THE THEODOLITE

A complete pattern of a general purpose transit theodolite with four open means of adjustment will form the subject of the treatment described in this article, the model selected being of the class most commonly used, the altitude level being (a) either on the telescope, or (b) on the clipping arm or vernier frame of the vertical circle.

In general, the theodolite may be said to embody *five fundamental lines*: (a) the vertical axis; (b) the plate level lines; (c) the line of collimation; (d) the transverse or trunnion axis; (e) the bubble line of the altitude (azimuthal) level, bubble lines being imaginary axes or tangents of undistorted bubbles (Fig. 19).

Conditions of adjustment. These are that (i) the plate level lines must be perpendicular to the vertical axis; (ii) the line of collimation must be perpendicular to the transverse axis; (iii) the transverse axis must be perpendicular to the vertical axis; and (iv) the altitude bubble line must be parallel to a horizontal line of collimation, (while it might be added (v) that the line of collimation should coincide with the axis of the telescope). The last, however, is a makers' condition, since two exact adjustments cannot be made with the "floating diaphragms" of theodolites, and in practice one would be approximate and the other exact.

The corresponding means of adjustment are the (1) plate level screws p, p, (2) horizontal diaphragm screws d, (3) standard nuts s, and (4) the altitude level tube screws b, b, (an approximate means being (5) in the vertical diaphragm screws).

Frequently the term "index error" is applied to maladjustment in respect to condition (iv), being the small vertical angle between the line of collimation and the horizontal bubble line of the altitude level. Strictly, this would be the *actual* index error, since apparent error would be eliminated by means of the clipping screws jj, the C vernier or microscope of the vertical circle having been set to zero with the relevant tangent screw.

So far as possible the treatment will be detailed from the practical standpoint, rather than with a view of outlining tests and problems of mere academic value, though, on the other hand, much of the technique

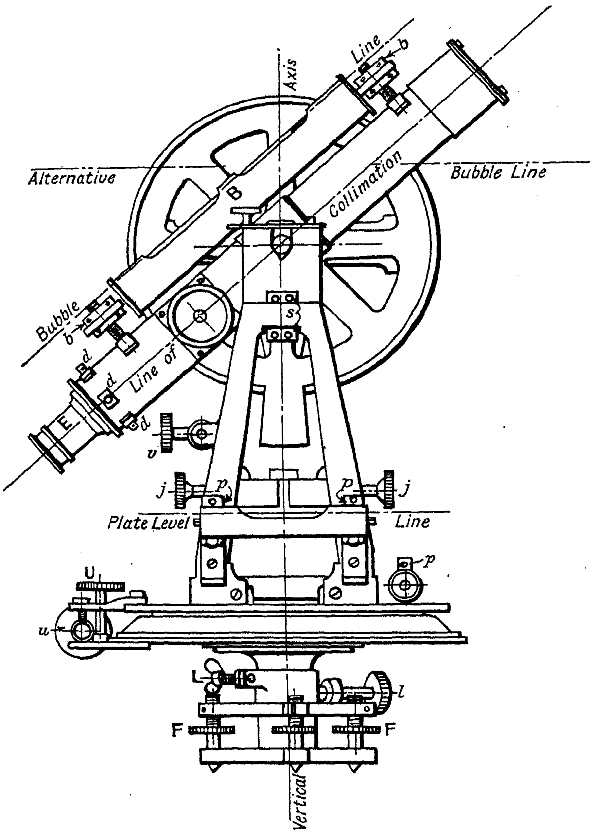


FIG. 19.

of the work is necessarily omitted. Wherefore, with this practical bias, no tests involving horizontal circle readings will be given, and, in consequence, this circle will be regarded as clamped throughout the entire procedure. In general, circle reading tests should be avoided so far as possible, since they introduce the errors of setting the verniers, together with the effects of eccentricity, which, though exceedingly small, will introduce some error to the relevant adjustments. Hence the two-peg test is preferred to " face reversals " in regard to the adjustment for (iv), particularly in the case of azimuthal levels attached to the telescopes.

For brevity, the abbreviations, *U.M.*, *L.M.* and *V.M.* will be used to denote respectively the clamp and tangent screw of the *upper* (*U*, *u*), *lower* (*L*, *l*) and *vertical* (*V*, *v*) motions, implying that the motion is

locked with the clamp and moved slightly by means of the corresponding tangent screw, either in setting the vernier, or in bringing the intersection of the cross-hairs on the image of the station mark. Likewise *F.L.* (*face left*) and *F.R.* (*face right*) will denote that the vertical circle is respectively on the left or the right of the observer, these positions frequently being known as the normal and reversed faces of a transit theodolite. Also the microscope should be understood in the use of the vernier throughout the procedure.

Example†. Describe how you would test and, if necessary, effect all the permanent adjustments of a complete transit theodolite in which the azimuthal level is carried (*a*) on the vernier plate of the vertical circle, or (*b*) on the telescope.

Preliminary. Select a piece of fairly level ground along which sights *PT*, *OT*, 200 to 250 ft., may be taken in opposite directions in the same straight line. A situation near a high building is desirable for the first theodolite station *T* if the instrument is not provided with a striding level. *Clamp the L.M. throughout the procedure.*

Adjustment of plate levels. (1) Level up the instrument at *T*, first to the plate level bubbles and finally to the altitude bubble, as follows, the process applying both when the altitude bubble is on the telescope or on the clipping frame. Set the *C* vernier of the vertical circle zero with the *V.M.*, and centralise the altitude bubble with the clipping screws *jj* when the eyepiece is over *one* tribrach screw. (2) Turn the telescope about the *L.M.* (that is, in azimuth) until it lies parallel to the other two tribrach screws, and centralise the altitude bubble with these if necessary. (3) Return the eye-piece end over the one screw, and, if necessary, correct the bubble with the clipping screws. (4) Now turn the telescope so that it lies end for end over the one screw ; and correct half the bubble error with this screw and half with the clipping screws. (5) Turn the telescope back through 90° and centralise the bubble solely with the two tribrach screws parallel to the telescope. (6) Repeat if necessary until the vertical axis is thus truly vertical. Set the plate level bubbles central by means of their adjusting screws *p*, *p*.

Adjustment of diaphragm. (1) Sight back *F.L.* on an arrow *P*, a distance of about 200 ft., using the *L.M.* (2) Transit the telescope *F.R.*, and sight a staff laid on the ground perpendicular to the direction *TO*, with the graduations towards the theodolite, the sight length also being about 200 ft. Note the staff reading *a*. (3) Turn the telescope about the *L.M.* (in azimuth) and sight the arrow *P*, *F.R.* Transit the telescope *F.L.* and note the staff reading *b*. If β is the error from 90° between the line of collimation and the transverse axis, then *ab* represents 4β, as indicated in Fig. 20. Measure $bc = \frac{1}{4}ab$ from *b*, the point *last sighted.* (4) Slacken the vertical diaphragm screws. Bring the vertical line of the

Two-peg test. (1) Find the true difference of level x from the difference $a \sim b$ of the balanced readings, keeping the C vernier at 0° throughout, and centralising the altitude bubble with the clipping screws. (2) Find the false difference of level y from the difference $a' \sim b'$ of the unbalanced readings. If $y = x$, the bubble line is parallel to a horizontal line of collimation. (3) Assuming that the adjustment is in error, $E = x \sim y$ in 4 chains, find $\frac{11}{10}E$ for 4·40 chains to give the reading f' on the distant staff A (Fig. 10). Set the cross-hair to the reading f' by means of the clipping screws, and centralise the bubble by means of its adjusting nuts b, b. Read d' on the near staff B, and if $f' \sim d' = x$ the adjustment is correct. Otherwise repeat the test and adjustment.

Face reversals. Frequently this method is given for sights on an elevated point, giving the index error as one-half the difference of the *F.L./F.R.* readings of the vertical circle readings. From the practical point of view, the following is to be preferred :

(1) Set the C vernier at 0° by means of the *V.M.*, and centralise the altitude bubble by means of the clipping screws. Sighting *F.L.*, read the horizontal hair on a staff (preferably at a target) at a distance of not less than 4 chains. (2) Unclamp both *V.M.* and *L.M.*, transit telescope, and again clamp the C vernier, but at the opposite zero. Sighting thus *F.R.*,

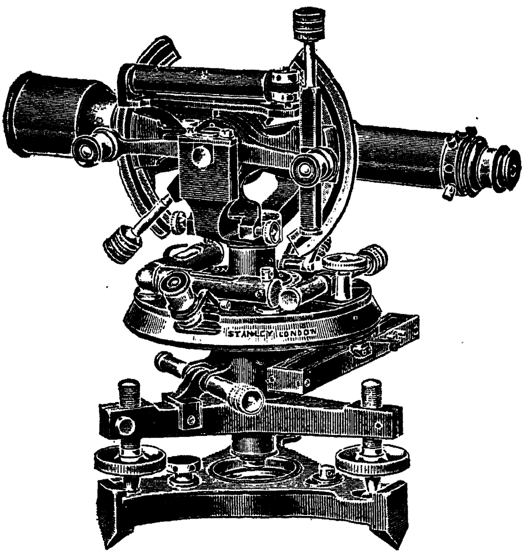


FIG. 22. Everest Theodolite.

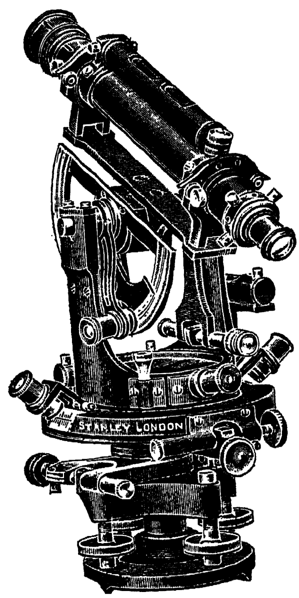


FIG. 23. Plain Theodolite.

read the staff. If the readings are not identical, unclamp *L.M.* and *V.M.*, transit *F.L.*, and reset the *C* vernier at the original zero. (3) Bring the cross-hair to the mean staff reading by means of the clipping screws, thus setting the line of collimation truly horizontal. Centralise the bubble of the altitude level by means of its adjusting screws *b*, *b*.

Example. Describe how you would test and, if necessary, adjust the following old pattern non-transit theodolites; (*a*) Everest's theodolite; (*b*) Plain or Y theodolite.

(*a*) **Everest's theodolite.**

This instrument has most of the representative features of the transit theodolite, but is characterised by two arcs instead of a vertical circle, the standards being so low that transitting is impossible.

Plate levels. Identical with that described for the transit theodolite (p. 39).

Diaphragm. This adjustment is effected by means of the horizontal diaphragm screws in the manner described for the transit theodolite (p. 40) after the double-sighting test has been made, usually *F.L.*, and then *F.R.* by removing the vertical and replacing it, reversed end for end, in the trunnion bearings. Adjustment is made with the vertical in its normal setting (see Note, p. 40).

Trunnion axis. Permanently established by the maker.

Altitude level. As in the case of the transit theodolite, apparent index error being eliminated by means of the clipping screws.

(*b*) **Plain theodolite.**

This instrument is essentially a Y level so far as the methods of adjustment are concerned, its construction normally being open to only one adjustment characteristic of the theodolite.

Plate levels. This is effected in the manner described for the transit theodolite, except that the *V.M.* is used instead of the clipping screws in the process of exact levelling-up.

Transverse axis, (ii) perpendicular to the line of collimation, and (iii) to the vertical axis. The use of the diaphragm screws is reserved for the following adjustment, while usually no provision is made for adjustment of the transverse axis.

Altitude level. Here two steps are involved in making the altitude bubble line parallel to the line of collimation: (*a*) Collimating, as described for the Y level, by sighting a point or reading and adjusting the centre of the cross-hairs to remain on this point by means of the diaphragm screws. (*b*) Making the bubble line parallel to the bearing surface of the collars by end for end reversals in the Y's, and making the half and half correction by means of the bubble tube nuts and the *V.M.* If index error is now evident, this should be noted, since usually there is no ready means of eliminating this. (See p. 20.)

QUESTIONS ON ARTICLE 5

1†. Detail concisely the tests and adjustments you would make relative to fitting the following parts to a 6″ micrometer theodolite which has been damaged in the field :

(*a*) Two azimuthal levels ; one attaching to the clipping frame and the other to the telescope.

(*b*) One micrometer microscope complete with lug for fitting to upper horizontal plate. (U.L.)

2†. Determine the so-called "index error" of a transit theodolite, that is, the inclination of the line of collimation when the azimuthal bubble is central and the verniers of the vertical circle are zero.

(*a*) Azimuthal level fitted to vernier frame or clipping arm.

(*b*) Azimuthal level fitted to telescope.

(*c*) When both (*a*) and (*b*) are provided.

State the objections to the use of the term in connection with transit theodolites. (U.L.)

3†. Following a mishap to a transit theodolite, the following parts are supplied by the makers :

(*a*) Spirit level in tube for attachment to top of telescope.

(*b*) Tubular diaphragm with lines on glass, giving vertical and horizontal lines and a stadia interval for a specified multiplier of 100.

Describe the methods by which you would test and verify the adjustments consequent to fitting these parts, also the way by which you would determine the exact value of the stadia multiplier. (U.L.)

4†. Describe concisely how you would test and, if necessary, effect all the permanent adjustments of a complete transit theodolite in which the azimuthal level is carried on the telescope. (U.L.)

5†. Describe concisely how you would test and, if necessary, effect all the permanent adjustments of a complete transit theodolite in which the azimuthal level is carried on the vernier plate of the vertical circle. (U.L.)

6. Explain how a theodolite is tested, and, if necessary, corrected, so that it may :

(*a*) have its axis of rotation vertical ; (*b*) have its trunnion axis horizontal ;

(*c*) be used to read vertical angles correctly.

Illustrate your explanations with suitable diagrams. (U.B.)

7. Describe the adjustments which should be carried out when new bubble tubes and diaphragm are fitted to a transit theodolite. For the vertical circle unit, assume that the alidade bubble is fitted to the vernier arm. (U.G.)

8. State the instrumental and other corrections which must be applied to the vertical circle readings of a theodolite, when the altitude of the sun is observed in connection with a position or azimuth determination. Explain briefly the reasons for each correction. (I.C.E.)

9. Describe, with sketches, how you would check the accuracy of the adjustments of a transit theodolite so as to ensure that when the telescope is turned on its horizontal axis, the line of sight traces out a vertical plane. (U.D.)

10. How would you test a theodolite to discover if the horizontal axis and the line of sight were perpendicular to each other? If adjustment of the line of sight be found to be necessary, how is the adjustment carried out? (U.D.)

11. In an examination of a theodolite it is found that, when the instrument is wheeled horizontally,

(*a*) the bubbles of the plate levels do not preserve a constant position in their tubes,

(*b*) the difference between the readings of the horizontal circle verniers is not constant.

Discuss these errors, and explain how you would prevent their influencing angular measurements. (U.D.)

12. Write a list of the "permanent" adjustments of the transit theodolite. Such an instrument is to be used for the measurement of horizontal angles. All the sights being approximately level, which of the above adjustments must be correct to ensure accurate results? Describe fully how you would check their accuracy. (I.C.E.)

13. Describe *one* good direct method and *one* good indirect method of obtaining contour lines.

Describe a quick and accurate method of adjusting a good class theodolite to enable vertical angles to be read correctly.

How is the value of a bubble division obtained? Explain the theory and application of the "bubble correction" to observed vertical angles. (U.B.)

14. Give a list of adjustments, both temporary and permanent, necessary before using a Vernier transit theodolite which is suspected of inaccuracy. The adjustments should be named in the order in which they are made.

How would you set right the adjustment of trunnion axis not being at right angles to the axis of the telescope? (T.C.C.E.)

15. A theodolite is in correct adjustment; what adjustments will you have to carry out if you replace:

(*a*) the diaphragm;

(*b*) the bubble vial of the vertical (or altitude) spirit level.

Outline briefly the method of carrying them out. (T.C.C.E.)

ARTICLE 6: ERRORS OF MALADJUSTMENT

The present article deals with those sources of error in angular measurement which arise from imperfections in the adjustment and construction of the theodolite and, though thus separated from imperfections of sight and touch, as inherent in the personal equation, it is obviously impossible in certain cases to exclude errors arising from purely personal sources.

Thus the errors to be considered arise from (I) Imperfect Adjustments and (II) Defects of Construction.

(I) The errors arising from imperfect adjustment of a theodolite are as follows :

(1) Vertical axis error, α. Axis not vertical in an observation, either from imperfect plate level adjustment, or settlement of the instrument.

(2) Lateral collimation error, β. Line of collimation not perpendicular to the trunnion, or horizontal axis.

(3) Horizontal axis error, γ. Trunnion, or transverse axis not perpendicular to the vertical axis.

(4) Vertical collimation error, δ. Tangent of altitude bubble not parallel to a horizontal line of collimation when the verniers or microscopes of the vertical circle read zero.

The above errors are often co-existent, wholly or in part in any given case.

II. The defects in construction are usually those of eccentricity and graduation.

The readings of opposite verniers or microscopes will usually differ slightly owing to one or more of three individual causes, the differences being exceedingly small but sometimes apparent with the fine degree of measurement possible with the micrometer.

(1) The microscopes may not be exactly 180° apart, as measured on a circle with the vertical axis as centre.

(2) The centre of this axis may not exactly coincide with the centre of the divided circle.

(3) Small irregularities may exist in the divisions of the circle, apart from backlash, etc., in the micrometer screw.

Of these the effect under (2) is the prime consideration, while those under (3) are partially involved with those of the personal equation.

(I) ERRORS OF ADJUSTMENT

(1) **Vertical axis error,** α. Apart from movement of the theodolite, error in this connection may not only arise from the fact that the plate level bubbles do not traverse, but that these may not be rigid or are sluggish, or are insufficiently sensitive.

Fig. 24 shows $A_0B_0C_0$ as the plane of a truly horizontal circle, while the plane of the actual circle is $A_0B_1C_0$, the greatest inclination, that of the vertical axis, occurring along OB_1. Here it is evident that the transverse axis will be horizontal when the telescope points in the direction OB_1, and that it will reach its maximum inclination when the telescope points along OA_0, the inclination being generally $\alpha \cos \theta$ when the telescope points in any direction, say OP, at a horizontal angle θ, read as θ' from OA_0 as zero.

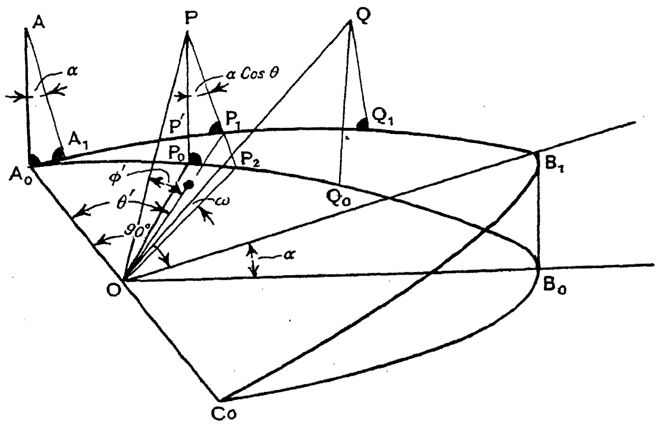


FIG. 24.

Thus the line of collimation will travel down a plane inclined at $\alpha \cos \theta$ to the vertical when the point P is sighted, and the error in position in the plane of the actual circle will thus be $P'P_1$, or P_0P_2 on the truly horizontal circle, the corresponding error being $\delta\theta$, which is practically the same for θ and θ'.

(*a*) *Horizontal angles.* Now $P'P_1 = PP_1 \tan(\alpha \cos \theta)$, since PP_1 is perpendicular to the actual circle, while PP_0 is truly vertical. But $PP_1 = OP_1 \tan \phi'$, while $OP_1 \,.\, \delta\theta = P'P_1$, ϕ' being the observed vertical angle. Whence

$$\delta\theta = \tan(\alpha \cos \theta) \tan \phi' = \alpha \cos \theta \tan \phi', \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

since $\delta\theta$ and α are very small quantities, and may be expressed in minutes or seconds, having a common factor in circular measure.

Now $\delta\theta$ in (1) is the error of sighting when the actual circle readings are θ' and ϕ', and since there must be two such sights in a measurement, say θ_1 and θ_2 at respective vertical angles ϕ_1 and ϕ_2, with errors $\delta\theta_1$ and $\delta\theta_2$, accordingly,

$$\Delta\theta = \delta\theta_2 - \delta\theta_1 = \alpha(\cos \theta_2 \tan \phi_2 - \cos \theta_1 \tan \phi_1), \quad \ldots\ldots\ldots\ldots(2)$$

$\theta_1 < \theta_2$ both being clockwise, measured from a zero along OA_0.

The sign of $\alpha \cos \theta$ does not change on reversing faces, and this process therefore will not eliminate the error. Hence the necessity of a truly vertical axis.

Note. Equation (2) expresses the maximum error, which becomes an absolute maximum when $\theta_1 = 0$ and $\theta_2 = 180°$, being then $\alpha(\tan \phi_1 + \tan \phi_2)$. Also if θ_1 and θ_2 are measured from OB_1 as zero, $\sin \theta$ must be written in place of $\cos \theta$. Generally, it is advisable to correct each reading rather than confuse signs by the use of formulae.

(*b*) *Vertical angles.* Case (i). When the azimuthal (frequently styled the altitude level) bubble is set central for each sight, as might be the case when the level is on the clipping arm.

Again, for a sight on P, where the true vertical angle is ϕ observed as

$$\text{Here}\quad \frac{PP_0}{PP_0 \sec(\alpha\cos\theta)}=\frac{\sin\phi}{\sin\phi''}\text{; whence }\sin\phi=\sin\phi''\cos(\alpha\cos\theta).\;..(3)$$

Writing $\sin\phi=\sin(\phi''-\delta\phi)$, and since $\sin\phi''\cos\delta\phi=\sin\phi''$ nearly,

$$\sin\delta\phi=\tan\phi''(1-\cos(\alpha\cos\theta))=\frac{\text{vers}(\alpha\cos\theta)}{\cot\phi''}\qquad(4)$$

$$\text{But}\qquad \sin\delta\phi=\sin 1''.\,\delta\phi,\quad\text{and}\quad \delta\phi=\frac{\text{vers}(\alpha\cos\theta)}{\cot\phi''\sin 1''}$$

Case (ii). When the azimuthal bubble is assumed to be correct (presumably along OA_0), as might be the case with a telescope level, the angle being measured from the plane $A_0B_1C_0$,

$$\frac{PP_0}{PP_2}=\frac{OR\,.\,\sin\phi}{OR\,.\,\sin(}\qquad=\cos(\alpha\cos\theta).$$

Hence the sine of the correct angle ϕ,

$$\sin\phi=\frac{\sin(\phi'+\omega)}{\sec(\alpha\cos\theta)},$$

where ϕ' is observed and $\omega=(\alpha\sin\theta)\sec(\alpha\cos\theta)$. ....(5)

Since $\alpha.\cos\theta$ does not change sign, the error will not be eliminated by using both faces of the instrument.

(2) **Lateral collimation error, β.** The effect of the error β is that when a point B' is observed at an actual vertical angle ϕ, the circles correspond with the sighting of B at an observed vertical angle ϕ', the corresponding error in the horizontal angle being $\delta\theta$, as in Fig. 25.

For each reading clockwise $\left\{\delta\theta \text{ is to be } \begin{matrix}\text{subtracted}\\ \text{added}\end{matrix}\right\}$ if of collimation for the face considered bears to the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right.$.

(*a*) *Horizontal angles.*

$$\tan\beta=\frac{BB'}{OB}=\frac{AA'}{OA\sec\phi'}=\frac{OA\tan\delta\theta}{OA\sec\phi'}.$$

Since $\delta\theta$ and β are small, their circular values may be substituted, leading to

$$\delta\theta=\beta\sec\phi'.\quad\text{..............................(5)}$$

(Thus the approximation admits the assumption of $\sin\beta$ for $\tan\beta$, sometimes given when the true value of ϕ is introduced.)

Since the measurement of an angle involves two sights, or pointings, the total error will be expressed by

$$\Delta\theta = \delta\theta_2 - \delta\theta_1 = \beta(\sec\phi_2 - \sec\phi_1), \quad \ldots(6)$$

where $\phi_1 < \phi_2$ are the observed vertical angles corresponding to θ_1 and θ_2, measured in the clockwise direction.

It is evident that the error is zero when both sights are horizontal or at the same altitude, and a maximum when one sight is horizontal, the magnitude of the error being determined by the other vertical angle.

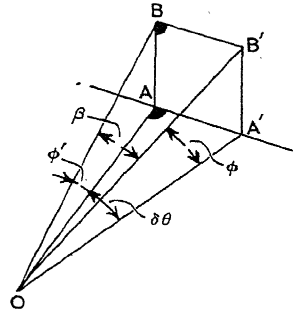


FIG. 25.

It also follows that the error is eliminated by transitting the telescope between successive rounds: the errors thus have the same magnitude but opposite signs, tan β changing sign in (5) with the face left and right observations.

(*b*) *Vertical angles*. Now $AB = A'B'$, and $OB \sin\phi' = OB' \sin\phi$.

But $OB = OB' \cos\beta$; whence $\sin\phi = \sin\phi' \cos\beta$.

Also $\sin\phi = \sin(\phi' - \delta\phi)$, where $\delta\phi$ is the error in the observed vertical angle ϕ', and $\cos\delta\phi = 1$ sensibly, hence (academically)

$$\sin\delta\phi = \frac{1-\cos\beta}{\cot\phi'}, \text{ leading to } \delta\phi = \frac{\tan\phi' \operatorname{vers}\beta}{\sin 1''} \text{ seconds.} \quad \ldots(8)$$

Except in precise high-altitude observations, the error is seldom appreciable, being only 0·23″ when $\phi' = 45°$ and $\beta = 5'$.

Since the sign of $\cos\beta$ is not changed, the error will not be eliminated in the mean of face left and right observations.

(3) Horizontal axis error, γ. The effect of the error γ is that when sighting a point B' at an observed vertical angle ϕ', the line of sight is brought out of the horizontal OA into the inclined plane AOB', the actual vertical angle being ϕ, corresponding to an error $\delta\theta$ in the horizontal angle, as in Fig. 26.

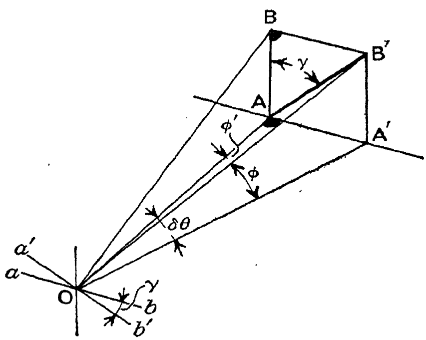


FIG. 26.

For each reading clockwise $\delta\theta$ is $\begin{Bmatrix}\text{added} \\ \text{subtracted}\end{Bmatrix}$ if the axis is high on the $\begin{cases}\text{left} \\ \text{right}\end{cases}$

(*a*) *Horizontal angles.* $AA' = BB' = OA \tan \delta\theta = AB' \sin \gamma$.

But $AB' = OA \tan \phi'$; whence $\tan \delta\theta = \sin \gamma \tan \phi'$ and, by the approximation in circular measure,

$$\delta\theta = \gamma \tan \phi' \text{ (min./sec.)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

(Thus the approximation admits the assumption of $\tan \gamma$ for $\sin \gamma$, frequently given when the true value of ϕ is introduced.)

Since, in measurement, two sights corresponding to clockwise horizontal angles $\theta_1 < \theta_2$ will be taken at vertical angles ϕ_1 and ϕ_2, the total error will be

$$\Delta\theta = \delta\theta_2 - \delta\theta_1 = \gamma(\tan \phi_2 - \tan \phi_1), \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

which becomes $\gamma(\tan \phi_1 + \tan \phi_2)$ when the angles are of opposite sign.

It is evident that the effect is neutralised when taking angles at the same elevation, also if the instrument is transitted between successive rounds. Normally error in the horizontal axis is more serious in its effect upon horizontal angles than lateral collimation error.

(*b*) *Vertical angles.* Assuming, as before, that there is no so-called index error, it merely remains to express the relation between ϕ' and ϕ as an error :

$$AB' = OB' \sin \phi', \qquad \text{and} \qquad A'B' = OB' \sin \phi;$$

but $\quad A'B' = AB' \cos \gamma$; whence $\sin \phi = \sin \phi' \cos \gamma$. $\quad \ldots\ldots\ldots(11)$

Writing $\sin \phi = \sin(\phi' - \delta\phi)$, where $\delta\phi$ is the error in the observed vertical angle; and since $\cos \delta\phi$ is sensibly unity,

$$\sin \delta\phi = \frac{\text{vers } \gamma}{\cot \phi'}; \quad \text{or} \quad \frac{\tan \phi' . \text{vers } \gamma}{\sin 1''} \text{ sec.,} \qquad .(12)$$

which is also the approximation given in the case of lateral collimation error.

Since $\cos \gamma$ does not change sign on reversing faces, the mean of such observations does not eliminate the error.

(4) **Vertical collimation error,** δ. Index error is a term frequently applied to the vertical angle between the horizontal line of a centralised azimuthal bubble and the line of collimation when the vernier or index of the vertical circle is set at zero. When these lines are both horizontal, as in an adjusted instrument, any reading on the circle is "apparent index error" i, which usually may be eliminated by conjoint use of the tangent screw and clipping screws of the vernier, or microscope, frame.

(i) *Azimuthal level on vernier frame.* Assuming that the vernier has been set at zero by means of the clamp and tangent screw of the vertical circle, and that the bubble line has been centralised with the clipping screws, then if sights, face left and right, are taken on an elevated point, half the difference of the circle readings (or pairs of readings) will give the actual index error whatever other errors exist, but the mean of the

readings will not give the true vertical angle unless other errors are known to be absent. If, however, the bubble is not central during these observations, the apparent index error will be given by $\frac{1}{4}(r_1+r_2)$, where $r_1=(a_1-b_1)$ and $r_2=(a_2-b_2)$, a and b being the readings of the ends of the bubble, the angular value following if the value of a bubble division is known. Half the difference of the observed elevations will be the actual index error plus or minus the apparent index error, or $\delta \pm i$.

(ii) *Azimuthal level on telescope.* Assuming that the apparent index error has been eliminated when the instrument was levelled up, face left, then if face left and right observations be made upon an elevated point, half the difference of the circle readings will give the actual index error.

Only in the case of azimuthal bubbles of low sensitiveness should the above process be regarded a sound basis of adjustment, since it involves the errors due to (*a*) eccentricity, (*b*) circle division, and (*c*) the three settings to a division; and in the author's opinion the single setting in the two-peg test is to be preferred, especially in the case of the more sensitive bubbles attached to telescopes.

Systematic investigation of errors. Since the errors α and γ may have the same effect, these will be considered together in relation to the bubble tests.

Let a_1, b_1 and a_2, b_2 be the readings of the same ends of the actual bubble, the differences of these being $r_1=(a_1-b_1)$ and $r_2=(a_2-b_2)$, while d'' is the angular value of a bubble division.

Bubble tests. (Errors 1, 3). Rotational reversals (about the vertical axis) with the azimuthal level will indicate if the axis is not vertical, the position of the greatest deviation giving the maximum error combined with the erstwhile constant bubble error. Thus if the same ends of the bubble be read for the position of maximum deviation and for 180° thereto, the axis error $\alpha=\frac{1}{4}(r_1-r_2)d''$, while the bubble error is $\frac{1}{4}(r_1+r_2)d''$.

(Vertical axis error is advisably eliminated at this juncture in tests, but it may have to be retained during certain observations.) Otherwise, the bubble error can be found by end for end reversals with the striding level; namely, $\frac{1}{4}(R'-R'')d''$ and duly corrected. If the instrument is now rotated about the vertical axis with the *adjusted* striding level in one position until the direction of maximum bubble deviation is found, then $R_1=A_1-B_1$, and upon further rotation through 180°, $R_2=A_2-B_2$, and $\alpha+\gamma=\frac{1}{4}(R_1+R_2)d''$, while $\alpha-\gamma=\frac{1}{4}(R_1-R_2)d''$. Whence α and γ are determinate. (If $R_1=R_2$, γ alone exists, the axis being vertical.)

The test may be checked by turning to 90° to the first position, where α will be zero, being in the vertical plane of the telescope.

(Although the vertical axis error α may be eliminated at this stage, occasions may arise where it should be retained and accounted for by applying the correction to the angles.)

Angular tests. (Errors 2, 3). If now face left and right readings of both the horizontal and vertical circles are taken to an elevated point, the mean of both pairs of readings will eliminate the errors of circle eccentricity, and the mean of the horizontal circle readings $\frac{1}{2}(h_1+h_2)$ will be the true horizontal angle, provided there is no error in the vertical axis. Likewise half the difference of these readings, $\frac{1}{2}(h_1-h_2)$, gives the error $\delta\theta$ due to the errors β and γ in the horizontal collimation and the transverse axis. But the magnitude of α is known, and if θ_0 is the angle from the position of no error, then the error in the horizontal angle $\delta\theta_1$ due to error in the vertical axis, is $\alpha \cos\theta_0 \tan\phi'$, where ϕ' is the mean observed vertical angle. Also the error $\delta\theta_3$ in the horizontal angle, due to error in the transverse axis, will be $\gamma \tan\phi'$, the sign following from the direction of the slope of the transverse axis. Hence the error $\delta\theta_2$, arising from error in horizontal collimation, may be calculated from $\beta \sec\phi'$; $\delta\theta_1+\delta\theta_2+\delta\theta_3=\delta\theta$. The true horizontal angle is the mean angle less the effect $\delta\theta_1$, due to error in the vertical axis.

(4) The mean of the two vertical circle readings, $\frac{1}{2}(v_1+v_2)$, is not the true vertical angle; one half the difference, $\frac{1}{2}(v_1-v_2)$, is the actual index error δ. One half the sum gives the correct angle plus the algebraical sum $\delta\phi$ of the errors from the three sources, and the individual errors may be determined from the expressions at the end of the following paragraph.

Miscellaneous tests. (Error 2). Either of the following might be used to determine the lateral collimation error β, the first being the "double sights" of the relevant adjustment. Hence sight back face left on an arrow P in level ground, transit face right, and read the staff at a when it lies horizontally at a distance D, say, 300 feet from the instrument. Turn the instrument about its vertical axis, and thus, face right, sight P; transit face left and read b on the staff. Then, since the vertical angles are equal or negligible, $\beta'' = \dfrac{\frac{1}{4}ab}{D \tan 1''}$.

The latter calculation can be avoided by sighting P face left as before with the A vernier zero, transitting face right, and sighting in an arrow Q; then turning about the vertical axis and sighting P face right; finally transitting face left and noting the vernier reading on exactly bisecting Q. One quarter of this angular discrepancy will be β.

Otherwise the horizontal angle between two points at the same level might be measured face left and right, giving the difference. Then $\beta = \frac{1}{2}\delta\theta \cos\phi'$, where ϕ' is the mean vertical angle, preferably zero.

(Errors 2, 3). Also it might be possible to sight a point on the horizon with the telescope in the vertical plane containing the vertical axis, and the difference of the face left and right readings would give $2(\beta+\gamma)$, of which γ is known from the striding level test.

(Error 4). It would be incorrect to assume that the mean of reversed vertical circle readings is the true vertical angle, yet one half the difference

of these is the index error. One half the sum gives the correct vertical angle plus $\delta\phi$ the algebraical sum of the individual errors in the vertical angle ϕ' :

$$(\alpha)\quad \delta^{1}\quad \frac{\text{vers}\,(\alpha\cos\theta)}{\cot\phi'\sin 1''}$$

$$(\beta)\quad \delta\phi_2 = \frac{}{\cot\phi'\sin 1''}$$

$$(\gamma)\quad \delta\phi_3 = \frac{\text{vers}\,\gamma}{\cot\phi'\sin 1''}$$

Preferably $\sin\phi = \sin\phi'\cos\beta$ (or γ)

(II) DEFECTS OF CONSTRUCTION

Defects of construction introduce errors of *eccentricity* and *graduation*.

If A is the reading of the microscope of that letter, and B the reading of the opposite microscope, $\pm 180°$, the difference is $B - A$, which is usually variable for different parts of the circle, the error $u = B - A$, including errors from the sources (1), (2), and (3) (p. 46).

(1) *Index Error.* From this cause there is a constant difference a which may be eliminated by adjustment of the microscopes (see p. 35).

(2) *Eccentricity.* Since the foregoing error is constant, it may be assumed that this has been eliminated in order to essay the error due to the second cause, eccentricity.

Let c be the centre of the divided circle and o the vertical axis and centre of the microscopes, being on a line between A and B in every position. Thus, for a position AB, the part of u due to (2) is represented by the angle AcA' in Fig. 27, where it is evident that the effect b varies for different positions of A and B, being zero when B reaches B_0 on the line A_0B_0. Then if m be the reading at B_0 and n the reading for any position B,

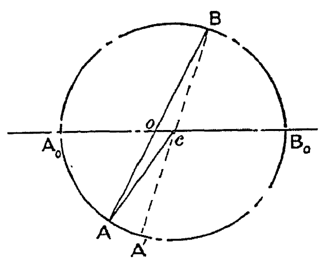


FIG. 27.

$(n - m) = BcB_0$, and if $oc = e$, the linear eccentricity, $Bc = R$, the radius of the circle, the ratio e/R being $e''\sin 1''$ very nearly ; and in the triangle ocB for any position of B, $\sin oBc = \dfrac{oc}{oB}\sin BcB_0$.

Let $b = AcA' = 2oBc$; then since oc is very small while oB is practically equal to Bc, $oc/oB = e/R$. Therefore

$$\sin\tfrac{1}{2}b = \tfrac{1}{2}b\sin 1'' = e''\sin 1''\sin(n-m), \text{ or } b = 2e''\sin(n-m).$$

It will be seen that if the telescope is swung through 180° from any position, the algebraical sign of b will be changed while b is numerically

unchanged. Therefore $2e'' \sin(n-m)$ is numerically the same for all pairs of positions 180° apart.

(3) **Errors of graduation and reading,** as already stated, are inseparable and are wholly irregular; and the error of any particular graduation is its variation from a position in an ideal system in which the differences from the actual divisions are as small as possible and are equally positive and negative. Therefore the average of all the errors for a large number of graduations at regular intervals round the circle will be zero.

But for any positions of the micrometers,

$$u=(B-A)=a+b+c=a+2e''\sin(n-m)+c.$$

Then if readings be taken at regular 10° or 20° intervals, a set of variable values will be obtained; and since c is assumed to average zero, and for every value of b there is an equal value of opposite sign, the mean of all the values of u is evidently a; so if a is subtracted from each value of u, $u-a=b+c$. But the last term changes to $-(b+c)$ for the opposite position of the microscopes, and so for n readings in opposite positions $(u-a)_\theta-(u-a)_{\theta+180°}=2(b+c)=2d$, say, or $d=2e''\sin(n-m)+c$.

Now it does not appear wholly reasonable to eliminate c from the values of d, whereas if d is put equal to b, and values for e'' and m are obtained, these values will be very nearly correct, since the mean value of c is zero and in summation from data will practically disappear. Thus is obtained a series of numerical values of d, each of which is $2e''\sin(n-m)$, and of these e'' may be the value to be found, n having a series of values, 10°, 20°, etc., while m is unknown.

Applying the method of least squares for readings taken at (say) 20° even intervals:

$$d_1=2e''\sin(0°\ \ -m)=2e''(\sin 0°\ \ \cos m-\cos 0°\sin m),$$

$$d_2=2e''\sin(20°-m)=2e''(\sin 20°\cos m-\cos 20°\sin m), \text{ etc.}$$

Multiplying these respectively (1) by sin 0°, sin 20°, etc., and summing, and then (2) by cos 0°, cos 20°, etc., and summing:

$$(1)\quad \Sigma(d\sin n)=2e''\{\cos m(\sin^2 0°+\sin^2 20°+\ldots\sin^2 160°)-\sin m(\sin 0°\cos 0°\ \ldots\ \sin 160°\cos 160°)\}.$$

$$(2)\quad \Sigma(d\cos n)=2e''\{\cos m(\sin 0°\cos 0°+\sin 20°\cos 20°+\sin 160°\cos 160°)-\sin m(\cos^2 0°+\cos^2 20°+\ldots\ \cos^2 160°)\}.$$

Now the second term of (1) and the first of (2) will cancel out between the 0° and 160° limits, while the $\sin^2\theta$ and $\cos^2\theta$ will add to 4·5 in either case. Thus

$$\Sigma(d\sin n)=N.\,e''\cos m,\ \text{ where } N=180/20=9\,;$$

$$\Sigma(d\cos n)=-N.\,e''\sin m.$$

Whence $\tan m = -\dfrac{\Sigma(d \,.\, \cos n)}{\Sigma(d \,.\, \sin n)}$, giving the angle of no *eccentricity*.

Now that m is known, e'' is determinate from

$$e'' = -\frac{\Sigma(d \,.\, \cos n)}{N \sin m}.$$

The reading m corresponding to no eccentricity being known, together with the value e'' of the eccentricity in seconds, the variable values of b are obtainable from $b = 2e'' \sin(n - m)$. Then if the several values of b are subtracted from the corresponding values of $u - a$, the error of graduation c for that angle may be found. It is evident that the values of b must vary in accordance with a definite law, depending upon the position of no eccentricity ; and the foregoing method will give the most probable result. The system really isolates the results, the values of c, as the most probable errors of graduation and reading.

QUESTIONS ON ARTICLE 6

1†. During observations at a station O a theodolite in perfect permanent adjustment was disturbed ; and, in order to avoid a re-setting, a striding level was placed upon the transverse axis of the instrument. When sighting a station P, the striding level read 2 and 10 divisions on the left and right respectively as viewed from the eyepiece, while on turning the telescope through a further clockwise angle of 64° 30′ the readings became 6 and 6.

Determine the true magnitude of a horizontal angle POQ measured in a clockwise direction from P, given that its observed value was 48° 30′ 20″ with angles of elevation of 6° 25′ and 15° 40′ to P and Q respectively, the angular value of a bubble division being 15″. (U.L.) [48° 30′ 21·6″]

2†. Explain with the aid of a diagram how error occurs in the reading of a horizontal circle when the trunnion axis is not mutually perpendicular to the vertical axis of a theodolite. Derive a formula for the correction involved.

Determine the true horizontal angle between P and Q in the following observation, which was made with a theodolite in which a division of the striding level corresponded to 15″, all other adjustments being presumably correct.

Object	Horizontal circle	Vertical circle	Striding level 1st pos. L	R	2nd pos. L	R
P	318° 20′ 00″	32° 24′ 40″	11·0	8·4	10·8	8·6
Q	15° 13′ 40″	5° 42′ 20″	11·2	8·2	10·6	8·8

(U.L.)
[56° 53′ 30·4″]

3†. Describe concisely how the line of collimation of a transit theodolite may be set truly perpendicular to the horizontal or transverse axis. State also how the method may be utilised in determining the angular error in this respect, and then show graphically the effect of such an error of 2′ upon the measurement of vertical angles between $\pm 90°$ in 15° intervals, in the case of a theodolite in which the azimuthal level is carried on the clipping arm. (U.L.)

4†. Describe concisely how you would determine the magnitudes of the following errors of adjustment in the case of a transit theodolite in which the azimuthal bubble is carried on the telescope and a striding level is provided.

(*a*) Axis not vertical when the instrument is levelled up with the plate levels.

(*b*) Line of collimation not perpendicular to the trunnion (or horizontal) axis.

(*c*) Trunnion axis not perpendicular to the vertical axis.

(*d*) Bubble tangent not parallel to horizontal line of collimation. (U.L.)

5†. An azimuth observation of a star was made with a transit theodolite known to be in perfect adjustment except for lateral collimation error, the following readings being observed :

Face	Horizontal circle		Vertical circle	
	A micr.	*B* micr.	*C* micr.	*D* micr.
Left - -	63° 30′ 10″	243° 29′ 30″	46° 59′ 30″	47° 00′ 20″
Right - -	243° 32′ 10″	63° 32′ 50″	46° 59′ 10″	46° 58′ 20″

Determine the true vertical angle, given that no correction for the azimuthal bubble is necessary. (U.L.)

[Since the mean of the two circle readings eliminates the errors of eccentricity :

Horizontally : Mean *F.L.* 63° 29′ 50″ ; mean *F.R.* 63° 32′ 30″.

Vertically : Mean *F.L.* 46° 59′ 55″ ; mean *F.R.* 46° 58′ 45″.

Index error $= \frac{1}{2}(59' \, 55'' - 58' \, 45'') = 35''$ subtractive from *F.L.* vertical angles. Since there is lateral collimation error, the mean sum is not strictly the true vertical angle ; and the difference of the horizontal circle readings is 2′ 40″ = 2 (collimation error at mean vertical angle 46° 59′ 20″) ;

$$\text{whence } \delta\theta = \tfrac{1}{2}(2' \, 40'') = \beta \sec 46° \, 59' \, 20'' \; ; \text{ and } \beta = 54{\cdot}6 \text{ sec.}$$

If ϕ' is the mean vertical angle and ϕ the correct value,

$$\sin \phi' = \sin \phi \cos \beta = \sin \phi, \; \cos \beta \text{ being } 1{\cdot}000.]$$

6†. Suspecting errors in the adjustment of his theodolite, a surveyor at station *O* sights a point *P*, face left, with the *A* vernier at 0°, transits and sights in a point *Q* which is on the same level as *P* and *O*. He then sights *Q*, face left, with the *A* vernier still at 0°, transits, and reads the circle as 359° 52′ for a sight on *P*. He next observes a station *M* in his survey, recording the following observation :

Face	Horizontal circle		Vertical circle	
	A	B	C	D
L - -	0° 0′	180° 2′	+ 46° 2′	226° 4′
R - -	179° 56′	359° 54′	313° 58′	133° 56′

State which of the adjustments are in error and determine the magnitudes of the errors, given that the axis is truly vertical. (U.L.)

[(1) Collimation error $\beta = \frac{1}{4}(360 - 359° \; 52') = 2'$.

(2) Mean horizontal angles : *F.L.* 0° 1′ ; *F.R.* 179° 55′.

Since two opposite pointings are involved, the error in the horizontal angle is $\frac{1}{2}(180° - 179° \; 54') = 3'$; but this may in part be due to trunnion error; hence

$\delta\theta = 3' = 2' \sec \phi' + \gamma \tan \phi'$, where ϕ' is the mean vertical angle 46° 3′

$= 2{\cdot}88 + 1{\cdot}0373\gamma$; giving $\gamma = 0{\cdot}12/1{\cdot}0373 = 0{\cdot}116' = 7''$.]

7†. Show in tabular form the possible faulty permanent adjustments of a transit theodolite, and indicate as far as you can the effects of the consequent errors upon a single measurement of a vertical angle, stating if reversing faces will eliminate these. (U.L.)

8†. An old theodolite gives widely differing readings on the A and B verniers at different parts of the horizontal circle; and you are required to determine the magnitude and direction on this circle of the maximum eccentricity. Describe concisely how you would proceed, giving the principles of the method employed together with a suitable tabular form for recording the notes. (U.L.)

9. You have been asked to use a theodolite, and it is suspected that the horizontal axis is not quite perpendicular to the vertical axis. In the absence of facilities for making adjustments to the instrument, show how you would arrange readings and field book entries so that this instrumental error could be eliminated. Select suitable figures to show the application of your method to vertical circle readings and to horizontal circle readings. (I.C.E.)

10. In the course of measuring horizontal angles of a triangulation survey, readings were taken of the vertical circle and of the position of the bubble in the striding level. Explain how you would correct the observed horizontal angles to eliminate the effects of horizontal axis dislevelment. (I.C.E.)

11. An arc of observations has been taken to six points distributed evenly round the horizon, and situated at varying elevations covering a range of vertical angles from $-30°$ to $+40°$.

The circle right and circle left readings show differences which vary from point to point, but on repeating the arc, show very nearly the same values as before.

Show how you would determine by inspection whether these differences are caused by an error of collimation, or an error in the horizontal axis, or are due to eccentricity, and develop a formula to give the effect of one of these errors on a horizontal circle reading.

Also explain in detail whether, and when if at all, these errors are likely to cause errors in the final result. (U.C.T.)

12. Describe the effects of non-perpendicularity of line of sight to trunnion axis and non-horizontality of trunnion axis in the various uses of a theodolite.

State the causes to which any discrepancies appearing in the table below may be due. Assuming the observations to be free of errors other than those above, obtain the corrected angle in azimuth between A and C.

Point	Right face		Left face	
	Hor. circle	Vert. circle	Hor. circle	Vert. circle
A	31° 4′ 5″	0° 0′	211° 4′ 5″	0° 0′
B	57° 22′ 10″	33° 0′	237° 23′ 20″	33° 0′
C	89° 44′ 25″	29°15″		

(U.G.)

[No collimation or index error. Presumably only trunnion axis error. 58° 40′ 20″ + 30·18″.]

ARTICLE 7 : TACHEOMETERS

Tacheometers may be classified in accordance with the principle they represent; namely, (1) Stadia or Subtense System, or (2) Tangential System :

(1) $D = l$ (2) $D = \dfrac{\quad}{\tan\phi - \tan\theta}$.

$$V = ks \,.\, \sin\alpha\cos\alpha + C\sin\alpha$$
$$= D\tan\alpha.$$

In these formulae D and V are respectively the horizontal distance and the vertical component; C is the additive correction $(f + c)$, which is zero in the anallatic telescope; k is the interval factor or constant multiplier, usually 100, which is f/i in external focussing telescopes, where f is the focal length of the objective and i the interval between the stadia lines of

the diaphragm ; α is the vertical angle in the subtense system, and ϕ and θ the pair of vertical angles inherent in the tangential system ; and s is the variable intercept between the stadia lines and the intercept corresponding to $\tan\phi - \tan\theta = 0{\cdot}01$ in Barcena's tangential method, s being fixed in the other tangential method.

The so-called (horizontal) subtense method, as occasionally used by topographers, merely introduces tacheometry through the medium of a theodolite, which is used to measure a number of times the small horizontal angle subtended by a short horizontal base. Here

$$D = \frac{s}{\theta'' \tan 1''} \quad \text{and} \quad V$$

Also the term " subtense measurement " is frequently applied to the determination of distances from the intercepts observed on a vertical staff on tilting the telescope through small vertical angles by means of the micrometer screw of certain modern levels : an operation formerly confined to the gradiometer.

I. **Subtense system.** Although accredited to William Green (1778), the subtense method is doubtless due to the illustrious James Watt. A Dane, named Brander, appears to have used a similar device in 1772. The principle is too well-known in regard to ordinary telescopes to necessitate a diagram. If parallel rays emerge from the stadia lines, interval i apart, they will converge at the principle anterior focus of the objective, and, thus crossing the optical axis, will intercept a distance s on a vertical staff when the telescope is horizontal. Thus if c (sometimes negligibly variable) is the distance from the vertical axis to the centre of the objective, then, for a horizontal telescope $(\alpha = 0)$, $D = (f/i)\,s + (c + f)$.

Anallatism means that the principal anterior focus is virtually moved to the vertical axis, so that in effect $C = 0$, and distances D are directly proportional to intercepts s. Formerly, the subtense lines were invariably horizontal, as they still are in British and American instruments. In these countries, the staff is held vertically, but on the Continent this was held perpendicularly to the line of sight, cancelling $\cos\alpha$ from the first terms of formulae (1), and thus admitting the use of ordinary trigonometrical tables in the reductions. In order to avoid the corrections for the foot of the inclined staff and the excess central line reading, the inclined staff has been largely superseded by a horizontal type, which is used in conjunction with *vertical* stadia lines. The staff is cumbersome with its tripod and vertical rod for the central line reading for tacheometric levels.

Intervals. Although the multiplier of *fixed* intervals is normally specified as 100, it is often not precisely this value, but sufficiently accurate for much ordinary work. Hence, in order to avoid correction tables, *adjustable intervals* were introduced. On the whole, these were not altogether satisfactory, being susceptible to variations in temperature and

requiring the use of points or wires—forms of " webbing " not popular with many surveyors. Incidentally, it should be borne in mind that $\frac{1}{1000}$ inch error in the interval means about 1 per cent error in distances with ordinary telescopes. The most reliable means of obtaining an exact multiplier is that of adjusting the additional lens of a true anallatic telescope, the form introduced by Porro in 1823.

Reductions. Stadia reductions may be made by means of (1) special and ordinary tables ; (2) diagrams and charts, and (3) special slide rules, though (1) alone should be used when a comparatively high degree of accuracy is desired.

Simplified reduction means the addition of a specially-divided vertical circle or arc, a graduated adjustable diaphragm, or a bar or other form of mechanism. Best known of all is **Beaman's stadia arc**, which introduces values of $\sin\alpha \, . \, \cos\alpha$ in increments of 0·01, 0·02, etc., with respect to the vertical component V. Here the importance of V is rightly emphasised, being independent of D, which when α exceeds about 3° is corrected by values $(1 - \cos^2\alpha)$ figured on the arc. Otherwise the arc or vernier arm is figured with $\tan\alpha$ in 0·01 increments, but the accuracy in V is then directly dependent upon D, which might otherwise have an approximate value in contour location, etc.

Automatic reduction. This suggests that D or V, or both, are read directly from the intercept s for all vertical angles. The best-known instrument is the **self-reducing tacheometer** of the late Dr. Jeffcott (Messrs. Cooke, Troughton & Simms). In the diaphragm, a fixed central point is used in conjunction with two movable points, which are actuated by an ingenious system of cams and levers, the telescope being anallatic. With this instrument D and V are simply read as $100\,s_1$ and $10\,s_2$, where s_1 and s_2 are respectively the intercepts between the fixed point and the upper and the lower points (see Fig. 28).

The author's self-reducing tacheometer (1913) had a rotating diaphragm with inclined stadia lines on glass, and these were moved towards the horizontal by a cam motion, wholly confined to the vertical. A special

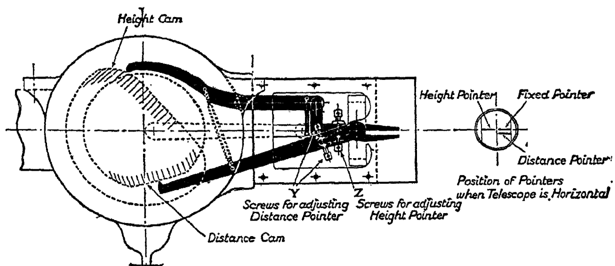


FIG. 28.

staff was necessary, since it was difficult to read the inclined lines on an ordinary pattern (see *Engineering*, April, 1915). A later experimental instrument embodied a pseudo-anallatic lens actuated by a cam motion.

In the writer's opinion there is no great future for automatic reduction, plausible as it sounds; and the simple Beaman arc will meet requirements wherever the method is employed expediently. Many misleading conclusions are based upon averages, whereas tacheometry suggests speed and not repetitions in order to extract mean values. The student must investigate for himself what errors are likely to result from such sources as one-thousandth of an inch variation in the interval, or the coincidence of an index on a circle differentially, not evenly, divided.

Moderate accuracy in tacheometry, particularly in regard to elevations, demands careful manipulation together with understanding and control of the sources of error.

Stadia micrometers. When the horizontal distance exceeds about 500 ft., with ordinary instruments the accuracy of stadiometry falls appreciably and becomes inferior to the tangential method, as used in the Bell-Elliot tacheometer. In order therefore to observe long distances, the targets of a fixed intercept, 10 ft. say, are observed with the aid of a stadia micrometer. This consists of a drum operating a screw, which moves the lines to the target base. Although the device may introduce the defects of adjustable intervals, its use is often warranted. In the author's opinion there are great possibilities in observing the movements of an anallatic lens on a millimeter scale (see p. 62).

II. **Tangential system.** The representative instruments of the fixed (10 ft.) base class carry a horizontal scale which is read with a microscope fixed at right angles to both the trunnion axis and line of collimation, the perpendicular distance from the axis to the scale

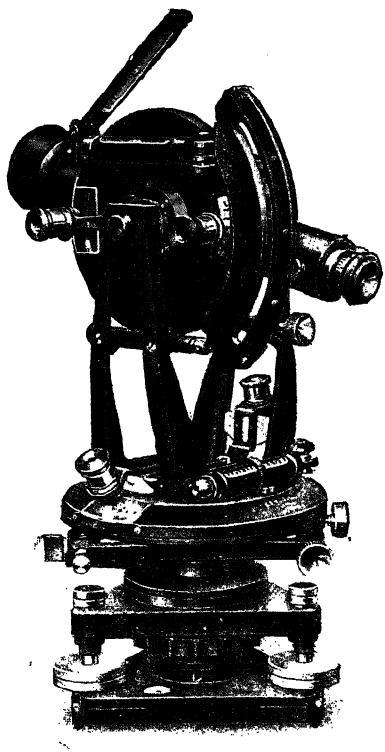


FIG. 29.

(6″) being known as the constant radius. With the aid of a simple micrometer the evenly-divided scale gives tangents to the equivalent of five decimal places : in the **omnimeter** a horizontal telescope corresponded to the number 50,000, the numerator being 150,000 s, while with the **Bell-Elliot tacheometer** natural tangents are read, reduction being facilitated with the aid of a table of reciprocals.

Barcena's method (variable intercept), though theoretically simple, was tried with no great success until the introduction of the **Szepessy tacheometer** (Messrs. E. R. Watts & Co.), the instrument rendering tacheometry as rapid as by automatic reduction (Fig. 29).

A scale of tangents of vertical angles with graduations corresponding to 0·01 and 0·005 is engraved on the fixed cover of the vertical circle, and, through the medium of prisms, these graduations are reflected through an aperture in the side of the telescope, so that the tangent scale is visible in the field of view with the image of the vertical staff, the horizontal and vertical lines of the diaphragm being also in focus. The horizontal line is brought to one of the numbered 0·01 divisions, and the intercept is taken between the *short* 0·005 divisions, immediately above and below. Thus $\tan\phi - \tan\theta$ is invariably 0·01, the horizontal distance D is $100s$, while the vertical component V is $D \tan\phi$, $\tan\phi$ being read as an even value at the horizontal line of the diaphragm (Fig. 30).

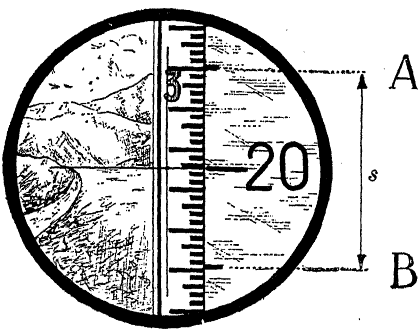


FIG. 30.

The term *tacheometry*, meaning rapid measurement, is sometimes used synonymously with *telemetry*, which is range-finding, the base s being at the observer and not on a distant staff.

Anallatic Telescope. If the focal lengths of the objective and the additional or anallatic lens are f and f' respectively, the distance between these being d, then formulae for the distance c between the objective and the vertical axis, and also for the interval i, may be derived to give a constant multiplier k, say, 100.

In Fig. 31, O is the objective and P the anallatic lens shown with reference to an intercept s on a staff held vertically at a horizontal distance D from the vertical axis V of the instrument.

By the stadia principle, the intercept appears as $a'b' = i'$ when the anallatic lens is omitted ; then

$$s/i' = f_1/f_2, \quad \text{or} \quad f_1 = \frac{f_2 \cdot s}{i'} \qquad .(1)$$

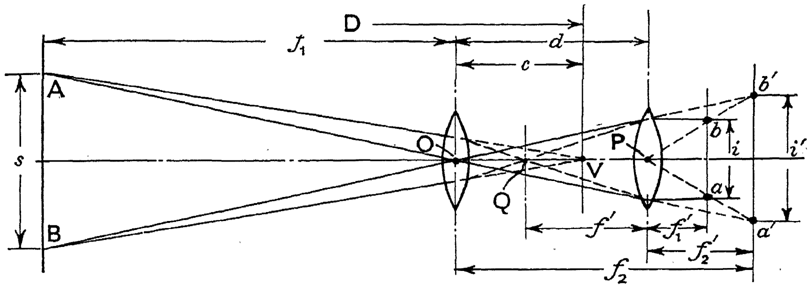


FIG. 31.

From the similar triangles aPb, $a'Pb'$,

$$i' = \frac{f_2'}{f_1'} \cdot i \,; \tag{2}$$

also by the conjugate relations for the anallatic lens P,

$$\frac{1}{f'} = \frac{1}{f_1'} - \frac{1}{f_2'}, \quad \text{or} \quad f_1' = \frac{f' f_2'}{f_2' + f'},$$

or since

$$f_2' = (f_2 - d), \quad f_1' = \frac{f'(f_2 - d)}{(f_2 - d) + f'} \cdot \ldots . \tag{3}$$

Whence i' in (2) becomes

$$i' = \left\{ \frac{(f_2 - d) + f'}{f'} \right\} i. \quad \ldots\ldots\ldots \tag{4}$$

Also by the conjugate relations for the objective O,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}, \quad \text{or} \quad f_2 = \frac{f f_1}{f_1 - f} \cdot \quad \ldots\ldots \tag{5}$$

Substituting for f_2 and i' in (1) leads to

$$f_1 = \frac{ff'}{i} \frac{s}{(f + f' - d)} - \frac{f(d - f')}{(f + f' - d)} \cdot \quad .. \tag{6}$$

Thus for the relation $D = f_1 + c = k \,.\, s$,

$$c = \frac{f(d - f')}{(f + f' - d)}, \quad k = \frac{ff'}{i(f + f' - d)}.$$

From the dotted lines of the figure it will be seen that the focus Q is conjugate to V for the objective O, and from this relation,

$$\frac{1}{f} = \frac{1}{c} - \frac{1}{d - f'}, \quad \text{or} \quad c = \frac{f(d - f')}{(f + f' - d)}.$$

Hence the vertex of the stadia angle is always at V in the vertical axis ; that is, Q is the principal anterior focus of P and focus of O for rays from A and B.

Error in the multiplier due to incorrect ruling of the subtense lines may be eliminated by adjustment of the anallatic lens.

Internal-focussing telescopes. If the optical relations of the internal-focussing telescope are assumed, these may be extended to the case of subtense lines in the diaphragm and a relationship determined between horizontal distances and intercepts read on a vertical staff.

Reference may be made to p. 4 and Fig. 1, where AB may be regarded as the intercept on a vertical staff and ab the actual interval i, $a'b'=i'$ being the virtual interval when the negative focussing lens P is omitted.

$$\frac{s}{i'}=\frac{f_1}{f_2}, \quad \text{or} \quad f_1=\frac{s}{i'}f_2;$$

and, on substituting the value of f_2 from (1), p. 4,

$$f_1=f\left(\frac{s}{i'}+1\right). \quad \text{.................................(5)}$$

Also $$\frac{i}{i'}=\frac{f_2'}{f_1'} \quad \text{and} \quad i'=\frac{f_1'}{f_2'}\cdot i=\frac{f_2-d}{l-d}\cdot i. \quad \text{..................(6)}$$

Thus (5) becomes $$f_1=\frac{f}{i}\,\frac{(l-d)}{(f_2-d)}\cdot s+f. \quad \text{.............................(7)}$$

If the vertical axis be placed a distance c behind the objective, the horizontal distance $D=f_1+c$. Hence if the telescope is to be anallatic, $D=k\,.\,s=100\,.\,s$ normally, and

$$\left\{\frac{\;}{i}\,\frac{\qquad}{f'}\qquad\qquad\right\} \text{ must equal 100.}$$

Theoretically, the telescope cannot be anallatic since the variables d and s are involved, but in practice this can be closely approximated to, though earlier patterns deviated appreciably from this ideal. In recent instruments the discrepancy is usually within the limits of error of observation in ordinary work, but in high-class surveys the exact deviations should be known, particularly if the instrument is to be used in tacheometrical levelling.

*Example**. Outline the principles of the stadia micrometer, showing how the multiplier and additive correction may be determined for a fixed intercept on a vertical staff. Sketch a well-known type of micrometer.

*Example**. Describe any one form of subtense micrometer, and show clearly how you would determine the value of the additive constant C in the case of a subtense micrometer in which there may be an initial reading of the micrometer head when the fixed and moving lines coincide, the focal length of the objective and the pitch of the micrometer screw being known. (U.L.)

Let s be the fixed intercept, say 10 ft., usually on a vertical staff, though this might be horizontal, or even normal to the line of sight.

Since the telescope will not normally be anallatic,

where the interval i is a function of the pitch p of the micrometer screw, and M the reading in turns of the drum when the subtense lines open to the intercept s, m being the reading when the fixed and moving lines are coincident. Thus m is effectively the index error, but might otherwise be some fixed initial value.

$$i=(M-m)p. \quad \text{.........(2)} \qquad D=\frac{f.s}{(M-m)p}+C. \quad \text{.........(3)}$$

If not otherwise known, f and C may be measured directly, while if p is not specified it may be found from the foregoing relations. In this connection it is necessary to measure the horizontal distances D_1 and D_2, and this should be done directly in preference to finding them from variable intercepts with the fixed stadia points fitted to some micrometers. Let M_1 and M_2 be the micrometer readings corresponding to the measured distances D_1 and D_2.

Then $$\frac{D_1-C}{D_2-C}=\frac{M_2-m}{M_1-m}, \quad \text{..............................(4)}$$

whence m is determinate.

If $M=0$ when the fixed and moving lines coincide, $m=0$ and $i=M.p$. Otherwise, m being known from (4), C and p may be found from the pair of observations:

where M_1' and M_2' are the corrected micrometer readings. But C is best measured directly.

In Stanley's micrometer, the pitch of the screw is $\frac{1}{100}$ inch, and five turns of the screw move the point over the space between any two fixed points. Single turns are shown on a star wheel and decimals on the micrometer head, each turn of the star wheel corresponding to 100. Thus

$$k=\frac{f}{M'p} \text{ reduces to } \frac{10000\times f}{N},$$

where $N=100M$ according to figuring (Fig. 32).

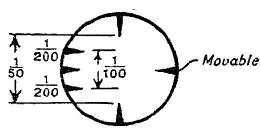


FIG. 32.

Example†. A theodolite is fitted with an ordinary telescope in which the eyepiece end moves in focussing, the general description being as follows:

Focal length of objective, $f=9''$; fixed

4†. In the event of a broken diaphragm in an anallatic telescope with a multiplier of 100, you are requested to determine the exact spacing of the lines on glass for a new diaphragm, the focal lengths of the objective and anallatic lenses being 9″ and $4\frac{1}{2}$″ respectively and the distance between the objective and trunnion axis 5″.

State clearly how you would insert the diaphragm, and give any other calculations that may arise. (U.L.)

[$i = 0{\cdot}07''$; $d = 7{\cdot}714''$]

5*. The following data were obtained in testing a stadia telescope fitted with a focussing tube, the telescope being level during the tests :

Distance (ft.) :	100	200	300	400	500
Objective to axis :	4·85″	4·82″	4·80″	4·78″	4·77″.
Staff intercept :	0·97	1·99	2·98	3·97	5·00 ft.

Focal length of objective 10″.

Determine the mean value of the multiplier and the additive constant, and state the percentage error that is introduced in horizontal distances by assuming the multiplier to have its specified value of 100. (U.L.)

[Mean $k = 100{\cdot}43$; 0·43 per cent error]

6*. In checking the stadia constants of a non-anallatic telescope in mountainous country, a photographic triangulation station P was occupied, while two similar stations Q, R, were utilised as stations for a *vertical* staff, the horizontal distances PQ and PR being respectively 505 ft. and 355 ft. by calculation.

From the following data determine the constants of the telescope, the eyepiece end of which moved in focussing :

Line : PQ ; Intercept, 5·09′ ; Vertical angle, $-5°\ 44'$.
,, PR ,, 3·61′ ,, ,, $+8°\ 6'$. (U.L.)

[$k = 99{\cdot}93$; $C = 1{\cdot}44'$]

7*. Derive an expression connecting horizontal distances and intercepts on a vertical staff in the case of an ordinary telescope provided with stadia lines.

State the error ΔD that would occur in horizontal distances D with an ordinary stadia telescope in which the focal length is 10 inches, the multiplier 100 normally, and the additive constant 1·2 feet, when an error of $\frac{1}{1000}$ inch exists in the interval between the stadia lines. (U.L.)

$$\left[\Delta D = -s\frac{f}{i^2} \times \Delta i = -s\frac{\frac{10}{12}}{(\frac{1}{120})^2} \cdot \frac{1}{1000 \times 12} = -s \quad \text{(Error equal to intercept)}\right]$$

8. Describe, with sketches, some form of tacheometer, in which by optical or mechanical means, or by a special system of graduation, the work of reduction of distance and differences of elevation is greatly simplified. (I.C.E.)

9. Deduce the fundamental formula for tangential tacheometry and describe developments of the system.

A hill-top known to be 2055 ft. above water level is observed from the opposite side of a lake, the altitude being 5° 10′. The angle of depression to the reflection of the same point in the lake surface is 8° 40′. Determine the horizontal distance from the instrument to the hill-top and the difference of level between them.

State the probable error (slide rule value) in horizontal distance if the probable error in angle measurement is $\pm 30''$. Tangents may be replaced by radian measure. (U.G.)

[18,244 ft. ; 1329 ft. ; $\pm 16{\cdot}9$ ft.]

10. Explain the principle of the tacheometer. A theodolite with eyepiece focussing is used for stadia measurements. Why have two constants to be determined for the instrument?

The observations below were made to determine the constants for a theodolite :

Distance of level staff from theodolite axis	Readings of stadia wires on staff	
	Lower wire	Upper wire
100 feet	3·62	4·61
200 feet	3·08	5·07

What are the values of the constants? (I.C.E.)

[$k=100$; $C=1$ ft.]

11. Obtain an expression for the distance D of a vertical staff from a tacheometer, if the line of sight of the telescope is horizontal.

How may the constants of a tacheometer be obtained?

From the following data, find the horizontal distance and reduced level of a point B, if :

(*a*) the staff is held vertically at B ;

(*b*) the staff is held at B normal to the line of sight of the tacheometer set up at A.

Tacheometer had a height of instrument of 5·05 ft. at A.

Vertical angle of elevation at A was 12°.

$$f/i = 100 \text{ and } f+d = 1{\cdot}3 \text{ ft.}$$

Readings on to staff at B were 6·22, 3·66, and 1·10 ft. Reduced level at A was 101·32 ft. (U.B.)

[(*a*) 491·15 ft. and 207·12 ; 502·11 (+0·76) ft. and 209·47.]

12. Discuss the relative merits of vertical and normal staff holding in stadia tacheometry.

Deduce the formulae for horizontal distance and elevation for the latter case. (U.D.)

13. Show that by introducing a lens at a fixed distance from the object glass in a theodolite telescope the distance E from an object to the instrument axis can be expressed by $E=kl$ where k is a constant and l the intercept cut off by the stadia hairs on a staff held at the point.

The focal length of the object glass is 110 mm.

The focal length of the anallatic lens is 90 mm.

The distance between the stadia hairs is 1·5 mm.

Calculate the distance between the two lenses and the distance between the vertical axis and the object glass if the multiplication constant is to be 100. Explain the effect of introducing an internal focussing lens instead of an anallatic lens. (U.C.T.)

[134 m.m.; 73·3 m.m.]

14. Explain in detail the differences between tacheometry and precision telemetry, including the purposes for which they are used, the essential field routine, and instrumental differences.

Illustrate the points brought out with references to tacheometric and telemetric equipment of well-known makers. (U.C.T.)

15. Distances may be found by optical arrangements (*a*) with a staff, (*b*) without a staff. Give examples of each type. Prove that and state when the error of length determination by these means is proportional to the distance and to the distance squared.

Describe in detail two methods of each case of finding distances using (i) a fixed intercept; (ii) a variable intercept.

Give details of the construction of two types of nomograms for reducing tacheometric observations. (U.C.T.)

ARTICLE 8: PHOTOGRAMMETERS

The title of this article is intended to comprehend surveying cameras generally, the primary divisions of which are (I) Ground Survey Cameras and (II) Air Survey Cameras, the former category including the photo-theodolite.

A survey camera may be described as a fixed-focus pattern provided with means of impressing " collimating marks " in the image plane, and suitably mounted so that the photographic axis is horizontal, vertical, or inclined at a known or determinate angle.

The above division follows from the demands peculiar to aerial cameras; in particular, lenses of wider angle, film magazines and mechanisms, special shutters, and exceedingly rapid films, apart from mountings and various accessories.

I. **Ground survey cameras.** There is no rigid distinction between surveying cameras and photo-theodolites beyond the fact that the latter are provided with horizontal circles to facilitate the orientation of the **picture traces**, which are the lines defining the intersection of the prints, or the equivalent positives, with the horizontal, or ground, plane. On the other hand, certain photo-theodolites can often obviate the use of a separate theodolite in various surveys.

Some cameras are (*a*) independent instruments, such as Prof. Dixon's "Surveyors' Camera"; some (*b*) may replace the theodolite in its tribrach, as was the case of Dr. Deville's model; while (*c*) others could replace the upper plate and alidade of the theodolite, as in the case of the U.S.A. Coast and Geodetic Phototopographic Camera.

The well-known Bridges-Lee Photo-theodolite (Messrs. C. F. Casella & Co.), (Fig. 33), is a combined camera-theodolite, fitted with vertical arc and auxiliary telescope, while Paganini's photo-theodolite was so constructed

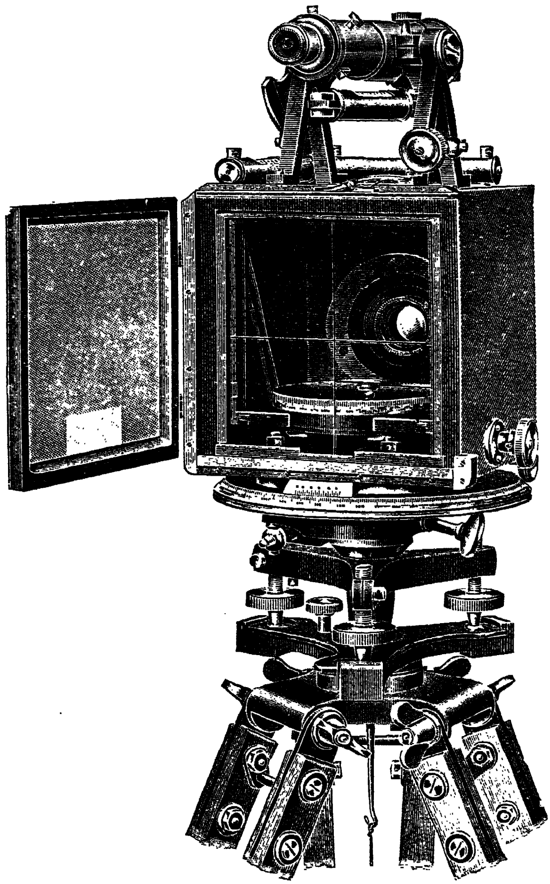


Fig. 33. Bridges-Lee Photo-theodolite.
(Messrs. C. F. Casella & Co.)

that the photographic axis was coincident with the axis of the telescope of the erstwhile theodolite.

Fundamental to the photogrammetric method, the following features are confined to the camera :

(i) Constant focal length f, the distance between the second nodal point of the lens and the image plane remaining constant for all exposures.

(ii) Adjustment ensuring perfect verticality of the photographic plate in the *normal* use of the method.

(iii) Means of impressing upon the negative the collimating marks, which fix the *horizon line hh*, and the *principal*, or *vertical, line vv*, the intersection of these lines being the principal point O of the perspective picture.

Except in the case of " obliques " in aerial photography, the collimating marks serve as controls rather than perspective elements.

Surveying cameras are so numerous in form and variety that only the essential features can be summarised briefly, as follows :

(1) **Levelling heads**, usually of the tribrach pattern, which are used in conjunction with cross-levels for ensuring verticality of the plates. It is desirable that the camera should be interchangeable with the theodolite without variation between the height of the photographic and optical axes.

(2) **Lenses**, which should comprehend angles upwards of 60°, and give, with great depth of focus, sharply-defined, undistorted perspectives, the photographic plate being illumined to uniform brightness. Among the well-known types may be cited Busch's pantoscopic, Dallmeyer's rapid rectilinear, Goerz' double anastigmatic, and Zeiss' anastigmatic.

Diaphragms, or stops, excluding rays from the marginal zones of the lens, should be used with technical understanding. These are specified by the ratio of the focal length to the diameter of the aperture, such as $f/32$, or by numbers in the same ratio as the area of the aperture, as No. 64, accordingly.

(3) **Impression marks**, fixing the positions of the horizon and principal lines, are afforded by contact of cross-hairs, points, or notches, with the photographic plate, an inner frame or case being moved by means of a head or screw. Deville employed notches on the inner edges of a rectangular frame at the rear of an inner case (Fig. 34), while in the Bridges-Lee model cross-hairs were used (Fig. 33).

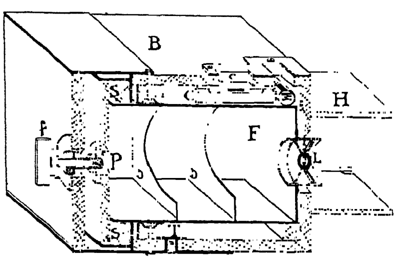


FIG. 34.

(4) **Plates** are usually preferred to films, the orthochromatic brands with a suitable colour screen, or ray

filter, being the most popular. Prints, of actual or enlarged size, are used in the graphical methods, while negatives are used in more precise work, and particularly in the stereo-photographic methods. The plates are slow in comparison with the films of aerial photography, exposures as long as one second being frequently employed.

(5) Special features include ground glass plates for viewing the landscape in certain models, transparent compass circles, pictured on the negatives, and in one instrument a transparent scale for reading directly the angular distances of points in the pictures.

(II) **Air survey cameras** may be grouped primarily into (i) single lens and (ii) multiple lens models, the latter being used in small-scale work rather than general mapping.

(i) **Single Lens cameras.** The features essential or peculiar to aerial cameras may be summarised as follows :

(1) **Mountings,** which are usually on gimbals, allowing for adjustment for tilt and drift, the unit being carried on a vibrationless support.

(2) **Lenses,** which are of the wide-angle design, the angular field ranging from 70° up to 95° in the ultra-wide types. Well-known lenses are the Ross Xpres (aperture : $f/4$, $f=7''$ for $7''\times 7''$ plates), and the Ross Wide Angle ($f/5\frac{1}{2}$; $f=4\frac{1}{2}''$), the Zeiss Topogon ($f/6{\cdot}3$; $f=10$ cm.) being a popular design.

(3) **Shutters,** which may be of (*a*) the *rotating disc variety,* as used extensively on the Continent and in the U.S.A., the types ranging from the quadruple blade to the multiple leaf, as interposed between the halves of certain Zeiss lenses ; (*b*) *slotted blind shutters* in the focal plane, which are giving way to (*c*) *louvre types,* composed of slats operating simultaneously between the lens and the focal plane. Louvre shutters are favoured in Great Britain, but the introduction of the ultra-wide angle lens is influencing the use of rotating disc or leaf shutters. In general, the time of exposure ranges from $\frac{1}{50}$ to $\frac{1}{150}$ of a second.

The Williamson " Eagle " cameras are generally used in Great Britain, the model supplied to the R.A.F. being commonly fitted with a Ross lens of $7''$ focal length, the photographs being $7''\times 7''$ (Fig. 35.)

(4) **Collimating marks,** corresponding to the points or notches impressed in the focal plane of ground survey cameras. Usually these are induced by the lines engraved on glass pressure plates, which also ensure that the film is in the correct position in the focal plane at exposure. Compressed air or suction is used in this connection. Another innovation is that of engraving the pressure plates with squares, or " reseaux ", of 1 cm. or 2 cm. side, in order that distortions over the photographs may be examined.

(5) **Films** in magazines are commonly used, sufficient for 200 to 300 exposures being contained in a spool. The spool and shutter are operated

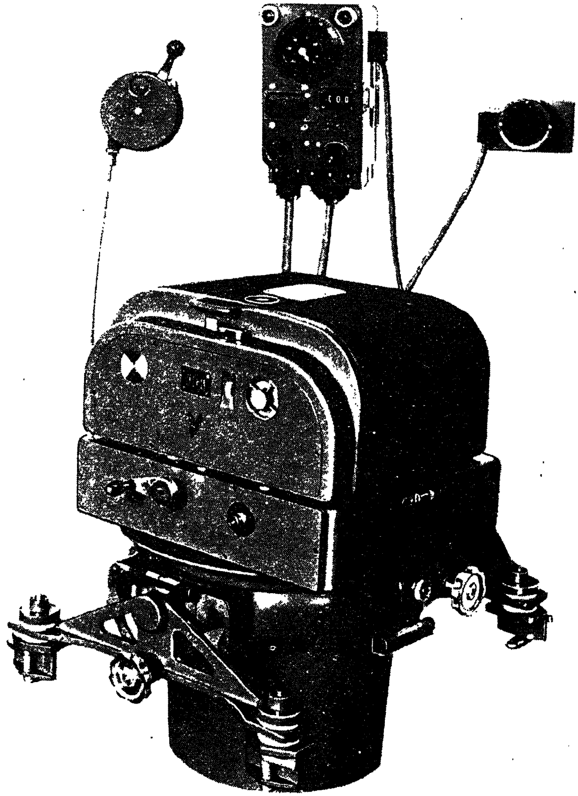


FIG. 35. Williamson "Eagle" Air Survey Camera.

electrically, but provision for mechanical drive is made for the contingency of electrical breakdown.

(6) **Recording instruments** usually consist of an altimeter, counter, and clock, the readings of which are pictured on the edge of the photograph, the seconds hand of the clock serving as a check on the exposure counter. When statoscopic equipment is carried, the relevant readings are photographed separately but simultaneously with the survey views. Mention may be made of a device for recording tilt, which, by gyroscopically-controlled light, defines the image of the plumb point at the instant of exposure.

(7) Accessories, including aiming sights, filters, etc. Aiming sights are used in vertical photography to ensure that all the area is embraced by the correct lateral overlap, the Aldis Aiming Sight being a well-known device. Filters of suitable colour for panchromatic plates are interposed between the lens and the film to reduce the lack of contrast that characterises photographs in which all the light has reached the film. The selection and use of filters is a subject of photographic technique, which also includes matters such as film and print distortion, emulsions, etc.

(ii) **Multiple cameras and multiple-lens cameras.** Several models were developed with the object of obtaining from obliques the results that would be obtained by vertical photography with what was then regarded an exceptionally wide angular field, for the use of obliques was considered the economical method of mapping on small scales on account of the limitations of scale in vertical photography. Fundamentally, a small-size central vertical was to be taken with a lens of short focal length, and around this a group of obliques of adjoining areas, the oblique axes being so related that an equivalent vertical could be obtained by rectification. An angular range of upwards of 120° was thus possible. Seven- and nine-lens cameras, such as those of Barr & Stroud and the U.S. Coast & Geodetic Survey, are well-known models.

However, the introduction of the ultra-wide angle lens has largely reduced the scope of these instruments, though there appears to be a field in small-scale mapping of under-developed areas where the differences of ground elevation are not great.

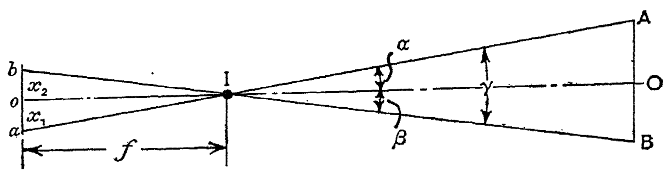


Fig. 36.

Focal length. Deville's method of determining the focal length, or distance line, of a surveying camera is concisely as follows (Fig. 36) :

(1) Set up the camera at a station I commanding a series of well-defined points, of which two can be selected near the horizontal limits of the plate, on or near the horizon line. Expose and develop the plate, and insert the horizon and principal lines. (2) Set up the theodolite at the station vacated by the camera, and measure γ the horizontal angle between the selected points A and B. Find the corresponding points a and b on the negative and measure the abscissae of these, x_1, x_2, from the principal line. (3) Calculate the value of f as follows :

$$\text{Tan}\,\alpha = \frac{x_1}{f}\,;\ \tan\beta = \frac{x_2}{f}\,;\ \tan\alpha\tan\beta = \frac{x_1 x_2}{f^2}.$$

But $$\tan(\alpha+\beta) = \tan\gamma = \frac{x_1/f + x_2/f}{1 - x_1 x_2/f^2}.$$

Whence $$f^2 - \frac{x_1 + x_2}{\tan\gamma}\cdot f - x_1 x_2 = 0.$$

Solving this quadratic, $$f = \frac{x_1 + x_2}{2\tan\gamma} + \sqrt{\frac{(x_1+x_2)^2}{4\tan^2\gamma} + x_1 x_2}.$$

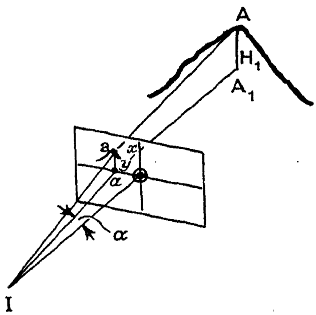


FIG. 37.

Paganini's method. (Fig. 37.) Here $f\sec\alpha = y\,\dfrac{D}{H_1}$;

but $x/f = \tan\alpha$, and $\sec\alpha = \sqrt{1 + \dfrac{x^2}{f^2}}$.

Whence $$f^2 = \frac{y^2 \,.\, D^2}{H_1^{\,2}} - x^2$$

Establishing principal and horizon lines. The following procedure applies to those types of surveying cameras which may be mounted so that either the long or short edges of the plates are vertical (Fig. 38).

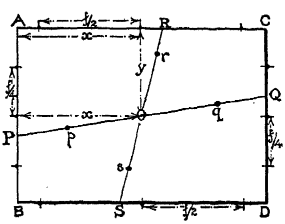


FIG. 38.

(1) Find with the level or theodolite two well-defined points P, Q, in the horizon, which, when photographed, will appear as p and q near opposite edges of one negative. Set up the camera in its *horizontal* position at the same height as the level, the plate being truly vertical. Expose and develop the plate, and scribe a line through p and q. (2) Now place the camera in its *vertical* position and repeat the operation,

if necessary, selecting two new points R and S, giving r and s as the horizontal line. Transfer r and s to the first negative by exact measurement along the edges of the plate, and obtain the principal point O at the intersection of pq and rs. (3) Measure from O the perpendicular distance to the edge AB of the negative, and set this off from the corresponding corners A and B of the rectangular opening to v and v' respectively. (4) Repeat the process with respect to the edge AC, thus obtaining h and h', which points are also notched with a file.

The foregoing method is particularly applicable to cameras of Deville's model, where notches are filed in the rectangular surround, and marks are made from hh' and vv' respectively equal to $f/4$ and $f/2$ in order to detect distortions in the prints.

Verifying principal and horizon lines. The following are the (*a*) *graphical* and (*b*) *analytical* methods of verifying the positions of the principal and horizon lines of photographic perspectives obtained with a camera which is rigidly attached to its levelling head.

Let A, B, etc., be the terrene points pictured as a, b, etc., and let A_1, B_1, etc., be the projections on the horizontal plane of the camera axis corresponding likewise to a_1, b_1, etc.

Select a view that will contain at least three stations, A, (B), and C, the distances and elevations of which are known with respect to the camera station I. Observe with the aid of a level that the picture will also contain two definite points at the same elevation, pictured as m and n, and, failing this, suspend a plummet so that its line VV will appear on the negative.

I. **Determination of principal line.** (*a*) *Graphically.* (1) Scribe a line pp on the negative parallel to the horizontal line through m and n (or perpendicular to VV); place a parallel-edged strip of paper $p'p'$ along this line and project a, (b), and c on to this strip as a, (b) *and* c (Fig. 39). (2) Lay this strip across the radials plotted in plan, and move it about until the projections fall upon the corresponding rays; a upon I_1A_1 and c upon I_1C_1, while the edge at o touches a circle of radius f, the tangency checking the solution (Fig. 40). (3) Mark o as the plan of the principal point O, and replace the strip on the assumed horizontal pp, and transfer the point O to pp. Scribe a perpendicular to pp through O as the required principal line vv.

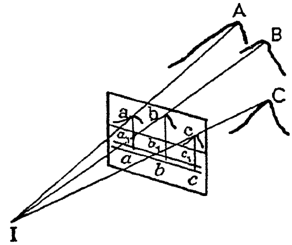


FIG. 39.

(*b*) **Analytically.** (1) Measure the distance $(x_1 + x_2)$ between a and c, where x_1 and x_2 are the abscissae with respect to an unknown principal point O. (2) Calculate x_1 from the equation between the angle γ observed between A and C with the theodolite and the focal length f (p. 76),

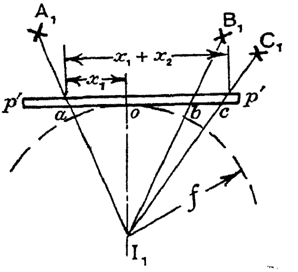


FIG. 40.

eliminating x_2. Measure x_1 on the negative and check by measuring $x_2 = oc$ (Fig. 40).

II. **Determination of horizon line.** (*a*) *Graphically.* (1) Measure the direction lines I_1a, (I_1b) and I_1c, also the sides I_1A_1, (I_1B_1), and I_1C_1 of the plotted triangulation. (2) Calculate the ordinates y_1 and y_2 of a and c respectively, the ordinate y_3 of b serving as a check (Fig. 39):

$$y_1 = a_1\mathrm{a} = AA_1 \frac{I_1a}{I_1A_1}; \quad y_2 = c_1\mathrm{c} = CC_1 \frac{I_1c}{I_1C_1},$$

where AA_1 and CC_1 are the known elevations of A and C with respect to the horizon of the camera station I. (3) Measure parallel to the principal line vv the ordinates y_1 and y_2 from a and c respectively, and scribe a line through their extremities a_1 and c_1 for the required horizon line hh.

(*b*) *Analytically.* (1) Calculate the lengths of the distance lines I_1a and I_1c from generally, and find the ordinates y_1 and y_2 of a and c from $d \,.\, \tan\theta$, where θ is the vertical angle of the pictured station, $\tan\theta$ being AA_1/I_1A_1 or CC_1/I_1C_1, with respect to the horizon plane of the camera station.

*Example***. An elevated point A, 124 ft. above the lens of a surveying camera, is pictured as a, 2·22″ to the left of the principal line and 1·12″ above the horizon line.

Determine the distance line, or working focal length, of the camera, given that the horizontal distance from the camera station to the point A is 620 ft.

Paganini's method. $f^2 = (1{\cdot}12)^2 \left(\frac{620}{124}\right)^2 - (2{\cdot}22)^2$

$= 31{\cdot}36 - 4{\cdot}93$, giving $f = 5{\cdot}14''$.

QUESTIONS ON ARTICLE 8

1*. The abscissae x_1, x_2 of two pictured points a and b are respectively 2·765″ and 2·260″ from the principal line of a photographic negative, while the horizontal angle between the corresponding terrene points A and B was 46° 19′ by theodolite measurement.

Ascertain the working focal length of the camera.

[$f = 2{\cdot}400 + \sqrt{5{\cdot}759 + 6{\cdot}249} = 5{\cdot}865''$]

2. Describe a standard type of photo theodolite, explaining with sketches how it is used, and show, from the photos, it is possible to obtain a contoured plan. (I.C.E.)

ARTICLE 9: MISCELLANEOUS INSTRUMENTS

Opportunity is taken in the present article of reviewing those instruments of lesser importance to the surveyor through the medium of relevant examples and questions. Thus the following instruments will be considered: (1) plane table; (2) sextant; (3) barometer; (4) compass, and (5) astronomical instruments.

(1) **Plane table.** This consists essentially of the alidade or sight rule and the table, the latter being a drawing-board mounted upon a tripod. Various sizes of boards are obtainable, from the simplest patterns with a thumb-nut clamp to elaborate forms with levelling heads and slow motions, and even with provision for carrying a continuous roll of paper. Likewise, alidades range from simple rules provided with sights to telescopic patterns, often complete with vertical circles. Usually a compass is provided for "setting" the board, setting being rough orientation, whereas correct "orientation" is the process of placing the board so that plotted lines are parallel to or coincident with the corresponding lines in the field.

Although there are three primary methods of using the table, **radiation**, **intersection**, and **traversing** (progression), the instrument is used for filling in topographical detail rather than for making entire surveys. "Intersections", the method commonly employed for inaccessible points, corresponds with the process of plotting from plate pairs in ordinary

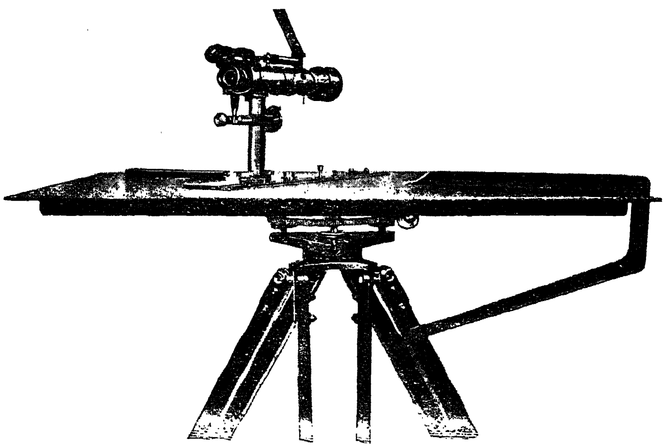


FIG. 41. Plane Table.
(Messrs. Cooke, Troughton & Simms.)

photographic surveying. Associated with the plane table are the well-known "three-point" and "two-point" problems, the former of which occurs in fixing soundings in hydrographical surveying. The three-point problem, in the case of the plane table, may be solved (*a*) mechanically, with the aid of a piece of tracing paper; (*b*) by trial, eliminating the "triangle of error"; and (*c*) graphically, the methods being based upon well-known geometrical theorems.

(2) **Sextant.** The sextant, as its name suggests, represents one-sixth of a circle, though the graduations, which are figured with twice their actual values, are carried somewhat beyond 60°, also on the *arc of excess* for convenience of reading the *index error*.

The instrument is made in two forms: (*a*) the **nautical sextant**, and (*b*) the **box sextant**, the sounding pattern being a variation of the former.

The nautical sextant consists of a sector-shaped limb; the periphery is a divided arc and the angle a centre for the index arm, which carries the wholly-silvered index glass. The other extremity of the index arm carries the vernier, which should read zero when the index glass is parallel to the half-silvered horizon glass, fixed to the limb near the arc. A clamp and tangent screw are fitted to the index arm, and usually a telescope is provided, though this may be replaced by a pinhole sight. Means of adjusting the index glass, the horizon glass, and the telescope are provided in better-class instruments. Sun shades are fitted to subdue the light in solar observations.

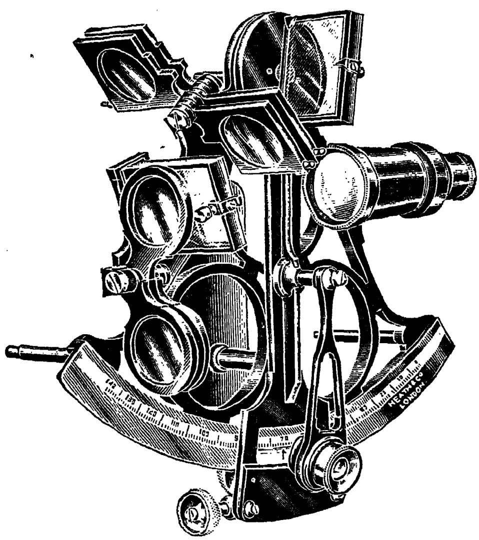


FIG. 42. Nautical Sextant.

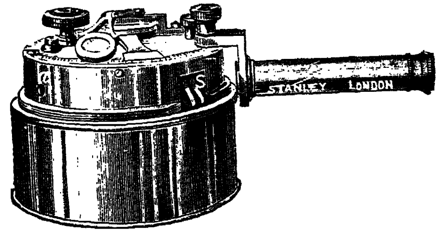


FIG. 43. Box Sextant.

The box sextant, a very compact instrument, embodies the above features, and is intended for land survey, but is used to a very limited extent. The nautical sextant is used for astronomical observations at sea, also extensively in marine surveying. Incidentally, the "three-point problem" in hydrography is solved mechanically with the station pointer, the process corresponding to the tracing-paper method on the plane table.

Peculiar to the sextant is the fact that angles other than vertical angles are measured in the plane of the observer's eye and the two observed points; and if the horizontal angle θ_g (as would be measured with a theodolite) is required, the oblique angle θ must be *reduced to horizon*, a process which involves the corresponding vertical angles α and β in the solution of the relevant spherical triangle—

$$\cos \theta_g = \frac{\cos \theta - \sin \alpha \cdot \sin \beta}{\cos \alpha \cdot \cos \beta}.$$

Vertical angles are measured on land with the aid of the *artificial horizon*, which is a mercurial mirror, or tray of oil, molasses, or water. Thus the image of the elevated point is brought into coincidence with the image viewed directly in the artificial horizon through the horizon glass, the observed angle being thus twice the altitude for very distant objects. At sea, the visible horizon is sighted and a correction for "dip" is applied.

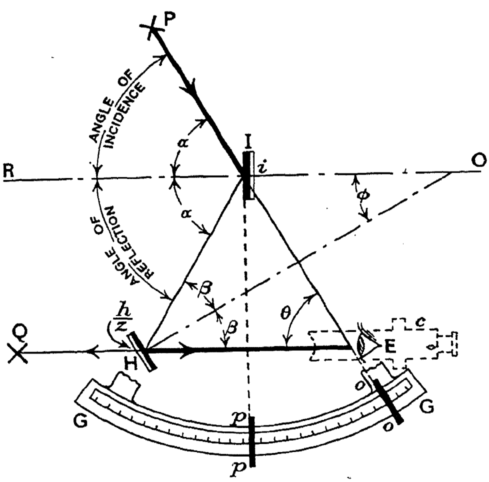


FIG. 44.

Optical principles. Let it be required to measure θ, the angle subtended at the eye E by two distant objects represented by P and Q in Fig. 44. The rays from P are incident upon and reflected from the index glass I at equal angles α to the normal RO, and thence the reflected rays

are incident upon the horizon glass I at an angle β to the normal HO, and, in reflection at β to the normal, are coincident with the direct rays from Q, as seen through the unsilvered part of the horizon glass.

From Fig. 44, $2\alpha = 2\beta + \theta$, and $\alpha = \beta + \phi$, and $\theta = 2\phi$, ϕ being the angle turned through by the normal RO and the index arm Ip. Also when the mirrors are parallel, θ is 0° and the arm occupies the position oo, and when the arm is in the position pp, θ being observed, the angle on the arc is $\phi = \frac{1}{2}\theta$.

Artificial horizon. The principle involved is that of measuring the angle subtended at the eye e by the object g and its image h as reflected from the surface of the 'horizon', the image being sighted directly in the mercury while the object itself is viewed by reflection from the index glass. The angle geh observed thus is twice the required altitude α (Fig. 45).

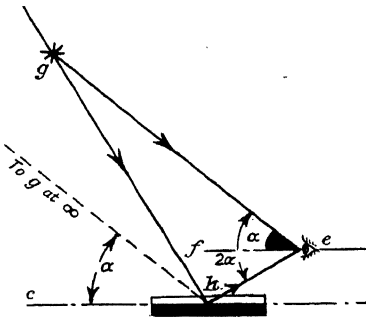


FIG. 45.

Assume horizontal lines ef and cd respectively through the eye and along the surface of the mercury. Then since the distances eg and hg are exceedingly great compared with eh, eg is parallel to hg; and the angle chg is equal to the angle feg, which is the required angle α. Since the complement of the angle of incidence chg is equal to that of the angle of reflection dhe, and dhe is equal to feh, the whole angle heg is equal to twice the angle feg; that is, twice the altitude α.

Example. Describe how you would verify the permanent adjustments of a nautical sextant. (U.L.)

Normally there are four adjustments of the sextant, and of these the third is frequently omitted, particularly when the horizon glass is otherwise in adjustment.

I, II. Setting the index glass and the horizon glass both perpendicular to the plane of the arc. III. Setting the index and horizon glasses parallel when the vernier reads zero (eliminating index error). IV. Making the line of collimation parallel to the plane of the arc. It should be noted

that considerable skill is required in effecting the adjustments of sextants, particularly for accurate observations, and of necessity many points in the technique are omitted from the following description.

(I) Set the vernier near the middle of the arc, preferably with the face uppermost on a table. Lower the eye almost to the plane of the arc, and looking obliquely into the index glass, view the adjacent portions of the arc, one directly and the other by reflection. If these appear as a continuous arc, the index glass is in adjustment; but if this is not the case, adjust the index glass by trial and error by means of the screw(s) i (Fig. 44).

(II) By sighting (*a*) a portion of the horizon, or (*b*) a well-defined point; for example, a star. (*a*) Holding the limb vertically, sight a smooth portion of the horizon, and bring its direct and reflected images into coincidence. Twist the limb bodily through an angle of about 20° and observe if the images still coincide, indicating correct adjustment. If this is not the case, adjust the horizon glass with the screws h. On land the ridges of roofs, overhead wires, etc., may be sighted. (*b*) Sighting the object, a star, say, sweep the index arm over the arc and observe whether the reflected image passes directly over the direct image, or overlaps it laterally. If the latter is the case, adjust to exact transit by trial and error, using the screw(s) h (Fig. 44).

A more precise adjustment can be made by solar observation, but the process involves technique.

(III) Bring the direct and reflected images into coincidence when sighting a star, a stretch of horizon, or the sun, or some well-defined terrestrial point. If then the vernier reading is not zero, the *index error* will appear on the arc proper or on the arc of excess. Eliminate by bringing the images into coincidence with the screw z when the vernier reads zero. Many retain the error rather than impair the adjustment of the horizon glass, which should be checked if adjustment is made.

(IV) (*a*) For terrestrial observations, lay the sextant, face uppermost, on a level surface, and place two objects of equal height (machined $\frac{1}{2}''$ nuts) on the extremities of the arc as temporary sights. Fix two pickets in line with these sights at distances of 15 ft. and 30 ft. from the instrument; and, sighting, direct an assistant to fix paper strips on the pickets to indicate the height of the temporary sights. Sight through the telescope, and again direct the fixing of strips. If the marks are equidistant on both pickets, no adjustment is required. Otherwise adjust the screws c of the collimating ring until the desired result is obtained (Fig. 44).

(*b*) For solar observations, sight with a pair of cross-wires parallel to the arc, and move the index arm until the images of two objects—sun and moon, or moon and star—more than 90° apart come into contact on the wire nearer the limb. Alter the position of the sextant slightly so as to bring the images on to the other vertical wire. If the contact still

remains perfect, no adjustment is necessary; but if they appear to overlap or separate, adjust the telescope by trial, using the screws of the collimating ring.

(3) **Barometer.** Although the mercurial barometer may be used as a standard at a base station, the portable form, the **aneroid** (=no liquid) is carried by the surveyor. Another instrument, also based upon variations in atmospheric pressure, the hypsometer, or boiling point thermometer, is used in rough determinations of altitudes. The **altimeter** used in connection with air-survey cameras is a form of aneroid, calibrated to give altitudes, but, being small, will not give absolute heights to within about 200 ft. Usually a **statoscope**, or differential aneroid, is used in addition to the altimeter, so that the variations can be more accurately determined.

Since the middle of last century the barometer has been used in determining altitudes in exploratory and pioneer work, also certain preliminary surveys. No considerable degree of accuracy was obtained, and the results were often disconcerting; though on the other hand much useful work was carried out where care and understanding were exercised in the use of a reliable instrument. Most of the difficulties arose from an indifferent knowledge of the physical bases of hypsometry, and aneroids, in particular, were often used with little regard for the initial conditions for which the instrument was calibrated. Tables with different initial assumptions were prepared by eminent physicists, and the surveyor, if ever investigating the subject, was bewildered with apparently conflicting data and corrections. Recently, considerable advances have been made in the subject, and, in 1935, a **standard atmosphere** was introduced as the basis of the War Office Aneroid Tables. Also, as a check on a faulty instrument, station and field batteries, each of three aneroids, are suggested for high-class hypsometry.

Mechanism of aneroid. The aneroid consists of a circular case with a glass cover, the base plate carrying the entire mechanism and the cover the dial. Fixed to the base plate is the vacuum chamber B, which is circular and corrugated and constructed of German silver, the walls being under 10 to 15 lb. per sq. in. of suction, and thus tending to collapse. Collapse, however, is prevented by the mainspring D, which is fixed to a bridge piece, the latter being so attached to the base plate that the mainspring is open to adjustment. Variations in atmospheric pressure induce pulsations in the vacuum chamber, and these are greatly magnified by the levers of the compensated gear, being finally read as pressures (in.) and altitudes (ft.) at the indicator on the dial. The gear consists of a compensated arm G operating through a crank axis H to a crank, which, pulling the chain Q, turns a drum and indicator, the motion being resisted by a hairspring (Fig. 46).

The altitude difference A with a barometer is $A = K(\log H - \log h)c$,

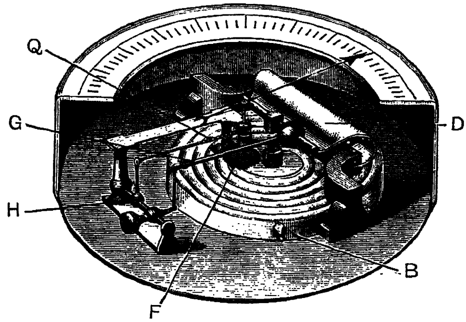


FIG. 46.

where H and h are the readings (inches) at the lower and upper stations respectively, and c the correction for intermediate air, $\left(1+\frac{T+t-64}{m}\right)$ with m ranging from 900 to 1020 according to the assumed initial temperature in the ratios of the absolute temperatures.

The multiplier K is $R\tau/g$, where R is the constant for air in absolute units, τ the *absolute* temperature in degrees Fahrenheit, and g the acceleration due to gravity, 32·19 ft./sec.^{2} near London. Thus with initial temperatures from 32° to 50° F. and from dry air to 50 per cent saturation, K will vary from about 60,384 to 62,759, which values are the common logarithmic modulus (2·303) times the height in feet of the hypothetical " homogeneous " atmosphere, approximately 5 miles.

The *corrections* usually applied to the mercurial barometer are as follows : (1) temperature of instrument ; (2) temperature of intermediate air, c ; (3) variation of g with latitude λ, $32{\cdot}09(1+0{\cdot}0053\sin^2\lambda)$; (4) change of g on a vertical ; and (5) height of lower station. In ordinary work only (2) is considered, the instrument itself being compensated for temperature, which is a different matter from the temperature of the intermediate air, though sometimes misconstrued.

Altitudes with the hypsometer are determined from Galton's Tables, or some such rule as $A=c(\beta T+T^2)$, where T is the reduction of temperature of boiling from 212° F. ; but many prefer to use the aneroid rule with h calculated from $29{\cdot}92-0{\cdot}586T$ inches.

Theory. Let a column of air exist at constant temperature between two points P and Q at altitudes A_1 and A_2 respectively. At some intermediate point in this column the pressure is p, while the density ρ is assumed to be constant instead of decreasing at higher altitudes as a compressible fluid. Then the following relation exists for an element of height δa, g being the acceleration due to gravity :

$$\rho g\,.\,\delta a=-\delta p, \text{ since } p \text{ decreases with increase in } a.$$

Also, for a perfect gas,

$$pv = R\tau, \quad \text{or} \quad \rho = \frac{p}{R\tau},$$

where R is a constant and τ is the absolute temperature Fahrenheit $(t+460°)$.

Thus $$\frac{\delta p}{\delta a} = \frac{-pg}{R\tau} \quad \text{and} \quad \delta a = -\frac{R\tau \,.\, \delta p}{g \,.\, p}.$$

Then if P and p be the pressures corresponding respectively to the lower and upper points,

$$\int_{A_1}^{A_2} \delta a = -\frac{R\tau}{g}\int_P^p \frac{\delta p}{p}; \qquad A_2 - A_1 = a = \frac{R\tau}{g}(\log_e P/p).$$

Since the pressures are represented by the barometric heights H and h at the lower and upper stations, accordingly,

$$a = \frac{R\tau}{g}\left(\log_e \frac{H}{h}\right) = k(\log_e H - \log_e h).$$

The multiplier k is the height in feet of the "homogeneous atmosphere".

Thus when $\tau = 492°$ F. $(t = 32°)$ and R for pure air is written in absolute units, namely, $53{\cdot}34 \times 32{\cdot}19$, the latter term being the value of g near London, $k = 26{,}220$ ft. $= 5$ miles nearly.

Reducing the last expression to common logarithms by multiplying by 2·3026,

$$a = K(\log H - \log h),$$

where $K = 2{\cdot}3026k = 60{,}384$ for pure air at 32° F., and 62,759 for 50 per cent saturation at 50° F.

Allowance for the temperature of the intermediate air is made by multiplying the altitude difference already determined by the ratio $c = \dfrac{\text{average abs. temp.}}{\text{basic abs. temp.}}$ or, less frequently, $\dfrac{\text{average abs. temp.}}{\text{abs. zero temp.}}$.

If the former is used with basic temperatures of 32° and 50° respectively,

$$c = \left(1 + \frac{T+t-64}{984}\right) \quad \text{and} \quad \left(1 + \frac{T+t-100}{1020}\right),$$

while with the latter, for a basic temperature of 32°,

$$c = \left(1 + \frac{T+t-64}{920}\right)$$

T and t being the temperatures at the lower and upper stations accordingly.

Example†. Draw a neat sectional elevation showing the mechanism of an aneroid barometer. Determine the altitude of a station B from the following data, assuming a constant multiplier of 60,384 for temperatures above an initial value of 32° F. and a divisor of 900 in the correction for the temperature of the intermediate air.

Reading at $A = 30{\cdot}28''$ at 9 a.m.; temp. 60° F.
Reading at $B = 29{\cdot}72''$ at 10.30 a.m.; temp. 46° F.
Reading at $A = 30{\cdot}33''$ at 12 noon; temp. 64° F.
The reduced level of A is 128·00 feet. (U.L.)

Variation in pressure in 3 hrs. (9–12) $30{\cdot}33'' - 30{\cdot}28 \quad 0{\cdot}05''$
" " " $1\frac{1}{2}$ " (9–10.30) $= 0{\cdot}025''$.
Probable reading at A at 10.30 is $30{\cdot}28 + 0{\cdot}025 = 30{\cdot}305''$.
Correction for temp. of intermediate air,

$$c = \left(1 + \frac{62 + 46 - 64}{900}\right) = 1{\cdot}0489,$$

with average temp. $T = 62°$ F.; $t = 46°$ F.
Alt. diff. $A = 60{,}384(\log 30{\cdot}305 - \log 29{\cdot}72)\ 1{\cdot}0489 = 535{\cdot}7$ ft.
Alt. of $B = 535{\cdot}7 + 128{\cdot}0 = 663{\cdot}7$ ft.

(4) **Compass.** Although a large variety of forms might be enumerated under this heading, it usually suggests (1) the prismatic compass, and (2) the surveying compass, as used in America. The former is still used in reconnaissance and pioneer work, though the theodolite (with its embodied needle) has superseded the compass, the characteristic features being retained in the mining dial.

The compass is too well known to merit discussion beyond recapitulation of a few technical terms and definitions. A magnetic meridian is the line assumed at rest by an unaffected needle, and the angle between the *true* and *magnetic* meridians is the **declination** at the place, the misleading synonym " variation " being often used in military and geographical text-books. Apart from the effects of magnetic storms, the *variations* in the *declination* at any place are of three kinds: (*a*) **secular changes** with the lapse of time, several centuries being involved in its full development (5′ to over 7′ per annum at Greenwich); (*b*) **diurnal variations**; more or less regular changes, attributed to the influence of sunlight (max. 10′ in any one day near London); (*c*) **annual variations**; cyclical changes in which the year is the period (less than 1′ at most places).

Bearings are recorded in either the **Quadrant System** (reduced bearings) 0°–90°–0°–90°–0° with the initial letter N or S and the terminal letter E or W; or the **Whole Circle System**, 0° to 360° clockwise. The former system facilitates the calculation of latitudes and departures.

Magnetic interference, or *local attraction*, is caused by the proximity of the needle to iron, steel, electric cables, nickel, etc. It is detected by the fact that in any line the back bearing (S α W) is not equal in magnitude to the forward bearing (N α E) ; that is, the bearings do not differ exactly by 180°. Work is carried out in the presence of magnetic interference by observing both backward and forward bearings, so that the included angle is determinate—a process tantamount to working " fixed " needle with the mining dial or theodolite. " Free needle " denotes that forward bearings only are taken, and the angles (presumably unaffected) are not isolated by reference to sights in both directions at any one station.

(5) **Astronomical instruments.** Although the bulk of the observations in field astronomy are carried out with a theodolite provided with axis illumination for night observation, there are two well-known instruments apart from chronometers and accessories—the solar attachment and the prismatic astrolabe.

The solar attachment. Although not generally favoured by British surveyors, this instrument is used by many, particularly Americans, with considerable success in determining azimuth and latitude whenever accuracy is subordinate to speed. The best-known patterns are those of Burt and Saegmuller, both American instruments.

The function of the attachment is that of giving a mechanical solution of the " astronomical " triangle *SPZ* (p. 262), the data of which are invariably the altitude α, and the declination δ, the third element being, as required, the hour angle ω, the latitude λ, or the azimuth A. Special tables are used for the refraction correction, which does not refer to refraction in altitude, but in declination, being reduced to a plane perpendicular to the celestial equator, usually in accordance with Chauvenet's formula.

The attachment consists effectively of two primary parts : (*a*) the *hour circle*, and (*b*) the *declination arc*, the *latitude arc* being formed by the vertical circle of the theodolite to which the instrument is attached by its polar axis to the centre of the telescope.

(*a*) The hour circle, the axis of which is styled the polar axis, is figured twice from I to XII, and is provided with a clamp and reading index. It is substantially the same in both patterns.

(*b*) The declination arc is divided to read minutes in its own plane, which also contains the polar axis. In the Burt attachment, a movable arm, which carries the vernier, is provided with a solar lens and an image plate at each extremity, while in the Saegmuller pattern an auxilliary telescope, appropriately shaded, is provided, and the declination arc is dispensed with, the declination being set off on the vertical circle of the theodolite and then reproduced by levelling the solar telescope with its attached spirit level.

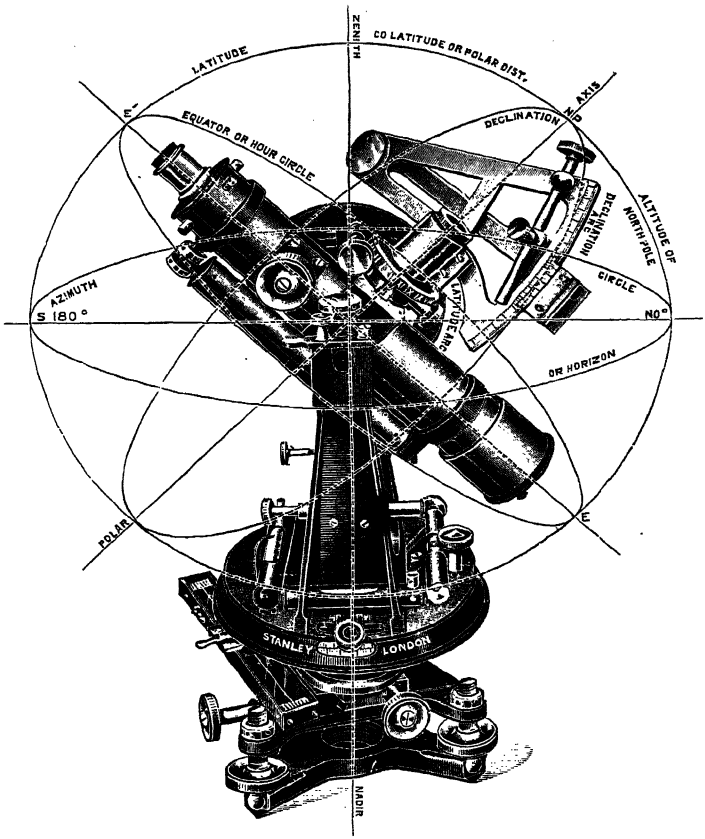


FIG. 47. Burt Solar Attachment.

The prismatic astrolabe. This instrument, a French invention, is used in observations for latitude, longitude, and time with reference to stars at a basic altitude of 60°. The prime feature is an equilateral prism P fitted in front of the objective O of a theodolite so that the face nearest the telescope is vertical, as shown in Fig. 48, where an artificial horizon A is shown in position.

Two images of a star are seen in the field of view: (*a*) *directly* from a face of the prism, and (*b*) *indirectly* by reflection from the artificial horizon and the other face of the prism. Since the first image is reflected once and the second twice, these images will move in opposite directions, according as the star is rising or setting. For example, if a star is setting

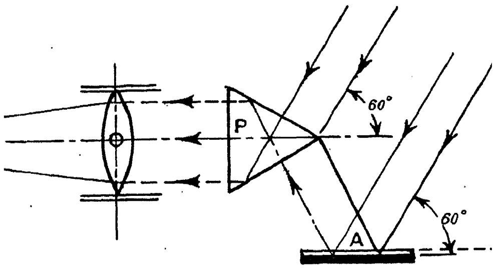


FIG. 48.

to the right of the observer, the images will appear to move towards the left, the indirect image uppermost, and at a given instant will be in the same vertical and will approach each other along inclined paths. When the altitude is exactly 60°, the images will coincide, and as the altitude decreases they will separate, the direct image now uppermost. Since the rate at which the images move is twice that at which a star would appear to move in the field of an ordinary telescope, the instant at which coincidence occurs can be determined much more accurately than with the webs of an ordinary telescope. Usually an electric chronograph is used for timing the coincidence.

In latitude observations, stars near the meridian are most suitable, while for time observations those near the prime vertical are preferable. Results are often worked out graphically by position lines. The result does not depend upon the accuracy of the angle of the prism, for if three observations are taken, the data are sufficient for solving for the three unknowns in the equations. Also, it is desirable, if possible, to make a number of observations in the four quarters of the heavens in order to reduce or eliminate the errors of uncertain refraction and of maladjustment of the instrument.

QUESTIONS ON ARTICLE 9

1*. Describe, with a neat sketch, the optical principles of the nautical sextant, and explain the theory of the artificial horizon in determining altitudes. (U.L.)

2*. Outline the optical principles of the nautical sextant, showing why the divisions on the arc are figured with double their actual values.

The following readings were taken with a nautical sextant, the direct and reflected images of opposite limbs of the sun being brought into contact :

On arc proper :	31′ 40″	32′ 00″	31′ 50″	32′ 00″.
On arc of excess :	31′ 20″	31′ 10″	31′ 10″	31′ 20″.

State the mean observed values of the sun's diameter and the index error of the sextant. (U.L.)

[Find mean reading on arc proper and arc of excess, viz., A and B respectively; then semi-diameter $= \frac{1}{2}(A+B) = 31' \, 33{\cdot}75''$, and index error $= \frac{1}{2}(A-B) = -18{\cdot}75''$].

3*. Draw a neat sectional view of a prismatic compass, naming the various parts of the instrument.

Define magnetic declination, and state why the synonymous use of the term " variation " is unqualified. Write a concise note on the three kinds of *changes* in the declination at any place. (U.L.)

4*. In a reconnaissance survey it is found that water boils at a station A at 202° F. with an air temperature of 34° F., and at a station B at 210° F. in an air temperature of 44° F., the altitude of B being 1030 ft. above sea level.

Determine the altitude of the station A :

(*a*) Using the simple hypsometer rule with a multiplier of 517.

(*b*) Using the aneroid rule with a multiplier of 60,384, and assuming the barometric heights to be $29{\cdot}92 \mp 0{\cdot}586\theta$ inches, θ being the difference of boiling point from 212° F.

[(*a*) 5261·0 ; (*b*) 5772·6 ft.].

5†. Describe the aneroid barometer, indicating the essential parts in a sectional view.

Show that the rule for determining altitude differences A is of the form

$$A = K(\log H - \log h)\,c,$$

where H and h are the barometer readings in inches at the lower and upper stations respectively, K a multiplier, and c a correction for the temperature of the intermediate air. (U.L.)

6†. Describe some form of solar attachment and explain concisely its use in an observation for (*a*) Azimuth, and (*b*) Latitude.

7*. Draw neat sectional elevations of the following instruments : (*a*) prismatic compass and (*b*) aneroid barometer, indicating and naming their essential parts. (U.L.)

8. Describe the marine sextant, explaining, with sketch, its optical principles. How would you check its adjustments? (I.C.E.)

9. Demonstrate the principle of the sextant by proving that the angle between the mirrors is half that between the objects brought into apparent coincidence.

Describe how you would find the value of the index error of an astronomical sextant. (I.C.E.)

10. Describe the nautical sextant and explain its principle. State its adjustments and how these are carried out. What is meant by the correction for dip, and what is the purpose and principle of an artificial horizon? In what respects does the box sextant differ from the nautical sextant? (U.B.)

11. Describe the essential components of a plane table outfit, and give the two usual methods of plotting a survey by means of the table.

Describe and explain the principle and use of the Beaman Arc.

Define " resection," and explain clearly the Two Point Problem and how it is solved. (U.B.)

12. (*a*) From a consideration of the physical laws governing the behaviour of gases, prove that the difference of height may be obtained from readings of barometric pressures by the formula :

$$h \text{ (in feet)} = \frac{2\cdot49}{0\cdot4343}\,\frac{13\cdot6}{0\cdot0013}\,\log\frac{B}{b}(1+\alpha t),$$

where $\alpha = \dfrac{1}{273}$.

(*b*) Reduced readings of a barometer in camp at half-hour intervals starting at 9.0 a.m. and ending at 12 o'clock were : 28·97″, 28·97″, 29·03″, 29·14″, 29·10″, 29·08″, 29·00″. A second barometer used in the field gave the following readings :

Time	Points visited	Reduced barometer readings	Reduced level
9.0 a.m.	Camp	29·03″	
10.45 a.m.	*A*	27·47″	2895 feet
11.20 a.m.	*B*	28·64″	
11·45 a.m.	Camp	29·10″	

Assuming a mean temperature of 77° F., find the reduced levels of Camp and *B*. (U.C.T.)

[952 ft. ; 1357 ft.]

SECTION II

ENGINEERING SURVEYS

INTRODUCTION

Although various problems, treated in other sections, will arise in connection with engineering surveys, this section is more especially reserved for what is understood as setting-out work, or field engineering. Thus the notes will be confined to the surveying and levelling operations in connection with the preliminary or Parliamentary estimates and the setting-out work in connection with the final or location surveys. Incidental to all these will be the location and use of contours, and the calculation of areas and earthwork volumes.

ARTICLE 1: EARTHWORK VOLUMES

Generally earthwork volumes may be divided into (1) **horizontal**, and (2) **vertical solids.**

The former includes the more or less regular prismoids represented by the cuttings and embankments of railways and highways, also irregular solids as determined by a series of cross-sections, the areas of which are usually found graphically, or mechanically by means of the planimeter. The latter category includes systems of square truncated prisms 50 to 100 ft. side, which also serve as a basis of contour interpolation, particularly in regard to areas devoid of definite surface features. It also includes the volumes determined from the areas defined by contour lines, as in the case of valleys impounded as reservoirs, the areas of the contour " strata " being usually found with the planimeter or computing rule.

The examples following are those of the first category which are amenable to arithmetical calculation, the volumes being straight in the present connection. Except in the case of *hill-side* sections, the treatment introduces a fictitious "whole area A'", corresponding to a total depth D, which is the cut or fill depth plus the depth w/s of the formation triangle, which being uniform in area leads to the same subtractive volume by all rules.

Usually the average-end area rule, or trapezoidal rule, is employed in preliminary estimates, and the prismoidal rule in final calculations, prismoidal corrections being sometimes applied to volumes calculated by the former rule. These rules are respectively the bases of Bidder's and MacNeill's tables.

CALCULATION OF EARTHWORK

The various cross-sections occurring in practice may be included in the following five types: (*a*) *level across sections*, (*b*) *laterally sloping sections*, (*c*) *hill-side sections*, (*d*) *three-level sections*, and (*e*) *multi-level sections*.

Notation. Half formation width, w; centre-line cut or fill, d; level half-width, W; side width on ground with a natural uniform slope, or on ground otherwise irregular transversely, W_l and W_r; side slope or batter ratio, $s:1$, being s horizontally to 1 vertically; uniform lateral slope ratio, $r:1$ likewise, r being $\cot\alpha$, where α is the vertical angle of lateral slope.

For convenience of calculation the side slopes will be produced to meet on the centre line, making the areas include the grade or formation triangles POQ, as in Figs, 49, 50, etc. The corresponding areas will be styled whole areas, A', as distinct from actual areas A, while the depth of cutting or bank on the centre line will be styled the whole depth, D, as distinct from the actual depth d. Accordingly the depth of the formation triangle will be w/s, and its area w^2/s invariably when level across, while the relevant volume will be the same by *all rules*.

(*a*) **Level across sections** (Fig. 49). Here the side width

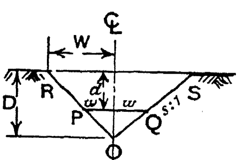


FIG. 49.

$$W = sD = s(d + w/s) = w + sd;$$

and the whole area $A' = sD^2$ ....(1)

$$= s(d + w/s)^2;$$

but area of formation triangle,

$$a = w^2/s;$$

hence true area $A = (2w + sd)d.$

(*b*) **Laterally sloping sections** (Fig. 50: $d > w/r$).

Side width

$$W_l = CT = W - R'T = W - s(RT)$$
$$= W - sW_l \tan\alpha = W - W_l\, s/r.$$

Whence

$$W_l = \frac{sD}{1 + s\tan\alpha} = \frac{W}{1 + s/r},$$

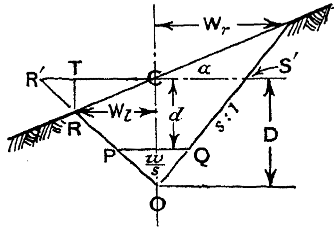


FIG. 50.

while likewise $\quad W_r = \dfrac{W}{1 - s/r}$;

and the whole area $\quad A' = \tfrac{1}{2}D(W_l + W_r) = \dfrac{sD^2}{1 - s^2/r^2} = \dfrac{A'}{1 - s^2/r^2}$. ....(2)

Thus if the areas are calculated as for ground-level across, the actual whole area can be found by means of an isoplethic diagram, giving

$$k = \frac{1}{1 - s^2/r^2}$$

for values of r or α, s being 1, $1\frac{1}{2}$, etc.

Further, if the longitudinal sections are plotted and an apex grade line is drawn w/s ft. above or below formation, the whole areas A' in (1) can be found on the slide rule, the factor k being applied in the case of (2).

(*c*) **Hill-side sections** (Fig. 51 : $d < w/r$). Since in practice the side slope ratio s for cuttings will frequently vary from s', the ratio in fill, it is advisable to deal with the areas separately as A_l and A_r, apart from considerations of expansion allowance. Thus the treatment is not simplified, as in the preceding cases.

Let h and h' be the depths QS and $Q'S'$ respectively.

Then

$$h = (w + x + sh)\,1/r = \frac{w + x}{r - s}$$

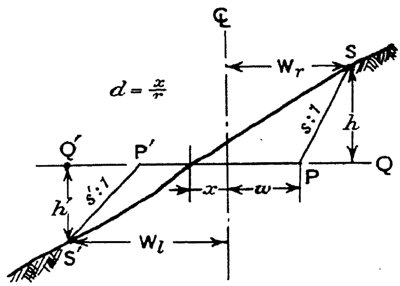


FIG. 51.

and $\quad h' = (w - x + s'h')\,1/r = \dfrac{w - x}{r - s'}$;

and $\quad W_r = w + PQ = w + sh = w + s(W_r + x)\tan\alpha$

$$= w + W_l s/r + sd = \frac{w + sd}{1 - s/r}.$$

$$W_l = w + P'Q' = w + sh' = w + s(W_l - x)\tan\alpha$$

$$= w + W_l\, s/r - sd = \frac{w - sd}{1 - s/r}.$$

Then $\quad$ area in cut, $A_r = (w + x)^2$.(3)

and $\quad$,, ,, fill, $A_l = \dfrac{(w - x)^2}{2(r - s')}$. .(4)

Incidentally, if the cut and fill areas are equal,

$$\frac{w+x}{w-x}=\sqrt{\frac{r-s}{r-s'}}=k, \quad \text{and} \quad x=w\frac{k-1}{k+1}.$$

(*d*) **Three-level sections.** This is a special case of the next category, but is given on account of its frequent occurrence when earthwork is calculated in the field, the side widths W_l and W_r being the slope stake distances.

$$A=A'-w^2/s.$$

(*e*) **Multilevel sections.** The following treatment is to be preferred to the usual method, in that it is not necessary to reduce the cuts and fills, the calculation of the area of the section being made directly from the observed staff readings (y) and horizontal distances (x). Here an origin of rectangular co-ordinates is assumed at the intersection of the line of collimation with the centre line, the plus directions being to the left and downwards.

Thus, in the "five-level" section of Fig. 52, x_0, y_0 refer to a staff reading on the centre line ($x=0$), x_2, y_2, the reading R at the slope stake

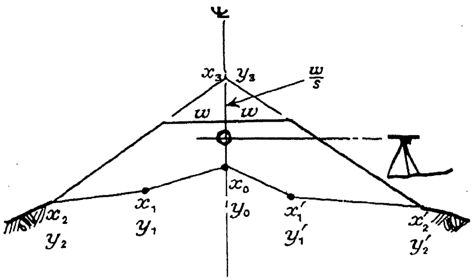


FIG. 52.

distance W_l, x_2', y_2', the corresponding values of R' and W_r, and x_3, y_3, the apex of the formation triangle ($x_3=0$) and $y_3=G+w/s=J$, where G (invariably plus) is $d+r$ for cuts, $d-r$ for high banks, and $r-d$ for low banks, with J plus for cuttings and minus for banks.

The Left, Centre, and Right readings are recorded as observed, and outside the extreme L and R readings, the values $J/0$ as though these were taken at the apex.

Then from the centre, running outwards left and right, the sum of the products of the abscissae, and the next outer ordinates is taken; thus, x_0y_1, x_1y_2, etc., and $x_0'y_1'$, $x_1'y_2'$, etc., terminating with J and W_l and W_r respectively, and giving a sum ΣU. Next, from the centre the

sum of the products of the ordinates and the next outer abscissae is taken likewise, thus, y_0x_1, y_1x_2, etc., and $y_0'x_1'$, $y_1'x_2'$, etc., terminating with R and O and R' and O respectively, and giving a sum ΣD. It is a simple matter of co-ordinate geometry to show that the whole area of the cross-section will be given by

..............................(5)

the true area A being $A' - w^2/s$.

In the accompanying form, the process is indicated with thick Up-strokes and thin Downstrokes corresponding to the U- and D-products.

	Left		C.L.	Right		
$\frac{x_3}{y_3}=\frac{J}{0}$	$\frac{y_2}{x_2}$	$\frac{y_1}{x_1}$	$\frac{y_0}{x_0}$	$\frac{y_1'}{x_1'}$	$\frac{y_2'}{x_2'}$	$\frac{J}{0}$
$\frac{-12{\cdot}38}{0}$	$\frac{5{\cdot}62}{27{\cdot}0'}$	$\frac{5{\cdot}40}{12{\cdot}4'}$	$\frac{4{\cdot}34}{0}$	$\frac{5{\cdot}60}{13{\cdot}3'}$	$\frac{5{\cdot}72}{27{\cdot}2'}$	$\frac{-12{\cdot}38}{0}$

Calculation of volumes. The degree of approximation with which the volumes are computed in railway surveys depends upon the order of those surveys: whether, for example, *preliminary surveys* or *final* or *location surveys*. In general, the areas may be regarded as approximating to those of more or less standard cross-sections in preliminary estimates, the transverse slope $r:1$ being determined from the contours of the map or by means of the clinometer, etc. In final estimates, it is often necessary to plot complex cross-sections, as in British practice, *or* to compute the areas from actual notes, as in the American method. Accordingly there will be a distinction between the rules by which the volumes are computed. Thus in preliminary estimates, the *trapezoidal* or *average end area rule* will suffice, while in final estimates, it will be necessary to employ the prismoidal rule, either directly, or by corrections applied to volumes determined from average end areas.

Further, the fact that a railway consists largely of curves becomes a matter for consideration in final estimates, and, in consequence, it will be advisable to divide the present discussion into (1) straight volumes, and (2) curved volumes, the latter applying only to final estimates.

Straight volumes. Average end area rule. In this rule, which corresponds to the trapezoidal rule for areas, it is assumed that the volume V consists of the length l multiplied by the average of the end areas A_1 and A_2, or $V = \frac{1}{2}l(A_1 + A_2)$, while for a series of sections l units apart,

$$V = \tfrac{1}{2}l(A_1 + 2A_2 + 2A_3 + 2A_4 \ldots A_n). \qquad \text{.................(6)}$$

Prismoidal rule. In this rule, which is the basis of Sir John McNeill's tables, it is assumed that the surface of the ground between any two vertical cross-sections is such that the volume contained is a prismoid, the end areas not necessarily being similar, but of any shape whatever, provided that the surface between their perimeters can be generated by straight lines continuous from one perimeter to the other.

The prismoid most nearly approximating to a railway cutting or embankment is an irregular truncated pyramid, such as that shown in Fig. 53.

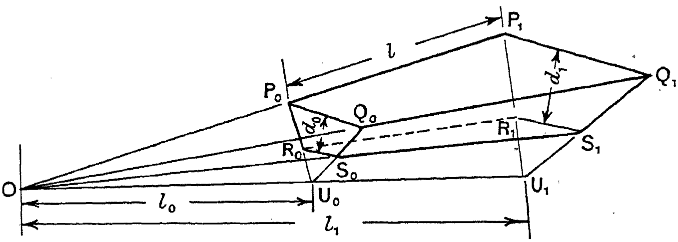


FIG. 53.

Let A_0 and A_1 be parallel sectional areas at respective perpendicular distances l_0 and l_1 from the vertex of the pyramid, l_0 being less than l_1. Then the content of the truncated volume will be $\frac{1}{3}(A_1l_1 - A_0l_0)$. But since the linear dimensions are proportional to the distances l_1 and l_0, the areas are expressed by the proportion

$$A_1 : A_0 :: l_1^2 : l_0^2, \quad \text{or} \quad A_1 = A_0 l_1^2/l_0^2. \quad \text{..................(i)}$$

Now the volume

$$V = \tfrac{1}{3}A_0(l_1^3/l_0^2 - l_0) = \tfrac{1}{3}(l_1 - l_0)\{A_0l_1^2/l_0^2 + A_0l_1/l_0 + A_0\}. \quad \text{.........(ii)}$$

Substituting $A_1 = A_0l_1^2/l_0^2$,

then $$V = \tfrac{1}{3}(l_1 - l_0)\{A_1 + A_0l_1/l_0 + A_0\}. \quad \text{....................(iii)}$$

Let A_m be the area of a section midway between A_0 and A_1;

then $$A_m : A_0 :: \{\tfrac{1}{2}(l_1 + l_0)\}^2 : l_0^2;$$

that is, $$A_m = \tfrac{1}{4}A_0(1 + 2l_1/l_0 + l_1^2/l_0^2).$$

Whence $$A_0l_1/l_0 = \tfrac{1}{2}(4A_m - A_0 - A_1). \quad \text{.......................(iv)}$$

Hence on substituting for A_0l_1/l_0 in (iii):

Prismoidal volume, $$V = \tfrac{1}{6}(l_1 - l_0)(A_0 + 4A_m + A_1)$$
$$= \tfrac{1}{6}(A_0 + 4A_m + A_1)l, \quad \text{.........................(7)}$$

where l is the distance between the end sections, normally 66 ft. or 100 ft.

The last expression holds both for whole areas such as $P_0Q_0U_0$ or true areas such as $P_0Q_0R_0S_0$, and is equally true when applied to end

sections with lateral slopes, not necessarily the same, the slope of the middle section in the latter case being the harmonic mean of those at the ends, while the middle depth remains the arithmetical mean of the end depths. Nor need the end sections consist of single surface slopes; for, in general, the volumes in question may be regarded as truncated irregular pyramids, and the prismoidal rule might have been derived in the first place with reference to such solids.

Although the theory assumes that each linear dimension of A_m, the middle area, is the mean of the corresponding dimensions of A_0 and A_1, it is occasionally taken incorrectly as the mean of these areas. If intermediate sections are not taken to determine A_m directly, this area may be found from the mean of the cut or fill (d or D) for A_0 and A_1.

Incidentally, if alternate sections in a series A_1, A_2, A_3, etc., are taken as the middle areas, $2l$ being written as the distance between the bounding sections:

$$V = \tfrac{2}{6}l(A_1 + 4A_2 + 2A_3 + 4A_4 + 2A_5 \dots A_n),$$
$$= \tfrac{1}{3}l\{A_1 + 4(A_2 + A_4, \text{etc.}) + 2(A_3 + A_5, \text{etc.}) + A_n\}, \quad \dots\dots(8)$$

which is Simpson's Rule.

Defence for the use of the prismoidal rule when the solid is not strictly a prismoid is to be found in its application as Simpson's Rule.

Frequently the volumes are calculated from average end areas, and a prismoidal correction is afterwards applied.

Prismoidal corrections. If A_1' and A_2' are adjacent whole areas, distance l apart, the volume by the *average end area rule* will be

$$V' = \tfrac{1}{2}l(A_1' + A_2'),$$

while the volume by the *prismoidal rule* will be

$$V' = \tfrac{1}{6}l(A_1' + 4A_m' + A_2'),$$

A_m' being the middle area—not the average of A_1' and A_2'.

Hence if the difference of these values be found, this may be applied as a "prismoidal correction", v, to the volumes determined from average end areas. Hence, generally:

$$v = -\tfrac{1}{3}(A_1' - 2A_m' + A_2') \text{ algebraically.} \quad \dots\dots\dots\dots\dots(9)$$

It is evident that this correction may be reduced to simpler terms in the case of the simpler sections, as follows:

(*a*) **Level across sections.** Since $A' = sD$, and $D_m = \tfrac{1}{2}(D_1 + D_2)$ for the middle area A_m', the expression in (9) reduces to

$$v = -\tfrac{1}{6}sl(D_1 - D_2)^2. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(10)$$

Also if true areas A_1, A_2, are given, the prismoidal correction will be the same since $(D_1 - D_2) = (d_1 - d_2)$, where d_1 and d_2 are the actual depths of the sections.

(*b*) **Laterally sloping sections.** When the transverse slope is uniformly $r:1$, the correction merely introduces $\frac{1}{2}(D_1+D_2)$ as the middle depth, and

$$v=-\frac{s}{6}\frac{l}{(1-s^2/r^2)}(D_1-D_2)^2=-\frac{s}{6}\frac{l}{(1-s^2/r^2)}(d_1-d_2)^2, \quad \text{.........(11)}$$

for whole and actual depths respectively.

If, however, the end slopes are different, r_1 and r_2 say, it would be simpler to use the formula for the prismoidal volume, the centre depth D_m or d_m being the arithmetical mean of D_1 and D_2 or of d_1 and d_2, and the centre slope r_m the geometrical mean of r_1 and r_2, namely:

$$r_m=\frac{2r_1r_2}{r_1+r_2}. \quad \text{...............................(12)}$$

(*c*) **Hill-side sections.** If the lateral slope r is constant, the correction for a pair of successive right or left areas, both cuts or fills, may be computed by substituting $d_m=\frac{1}{2}(d_1+d_2)$ in the general expression for the prismoidal correction; then

$$v=-\frac{lr}{12(1-s/r)}(d_1-d_2)^2.$$

Further digression on this case would be academic.

(*d*) **Three-level sections.** Since the sums of the side widths (W_l+W_r) will appear in the volume formulae, the whole widths $B=(W_l+W_r)$ may be introduced as B_1, B_2, etc., corresponding to whole depths D_1 and D_2, the depth at the middle section being $\frac{1}{2}(D_1+D_2)$; then by the rule in (9),

$$v=-\{B_1(D_1-D_2)-B_2(D_1-D_2)\}\frac{l}{12}=-\frac{l}{12}\{(B_1-B_2)(D_1-D_2)\}, \quad \text{...(13)}$$

which is the same when actual depths d_1 and d_2 are used.

QUESTIONS ON ARTICLE 1

1*. Derive expressions for the cross-sectional areas of three typical cases of cuttings along the straight centre line of a railway.

2*. State and prove a formula for computing the cross-sectional areas of a railway cutting, assuming the lateral slope of the ground to be r horizontally to 1 vertically, and the side slopes s horizontally to 1 vertically, d representing the central depth of cutting and w the half-formation width. (U.L.)

3*. Derive the so-called prismoidal rule for earthwork volumes, and indicate under what conditions it becomes identical with Simpson's Rule.

4*. Determine the prismoidal corrections applicable to volumes calculated by average end areas of the following types: (*a*) ground-level across; (*b*) ground sloping laterally; (*c*) "three-level" sections.

5*. A road is to be constructed on a plane hill-side, the centre line running at right angles to the line of steepest slope, which is 6 horizontally to 1 vertically. The formation width is to be 30 ft., level transversely, and the side slopes $1\frac{1}{2}$ horizontally to 1 vertically in cuttings and 2 to 1 likewise in embankments.

Find, to the nearest 0·01 ft., the depth of cutting in the centre line so that the cross-sectional areas of cutting and filling at a given cross-section are equal. (U.L.)

[$d = 0{\cdot}073'$]

6†. A straight level road is to be constructed on a plane hill-side with a lateral slope uniformly 9 horizontally to 1 vertically, the side slopes being likewise 1 : 1 and 2 : 1 in cut and fill respectively and the formation width 30 ft.

Calculate the total volume of earthwork in a length of 600 ft. :

(*a*) when the areas of cut and fill in each cross-section are equal ;

(*b*) when the total earthwork in each cross-section is a minimum, stating the volume of cut in excess of the fill in the latter case. (U.L.)

[(*a*) $x = 0{\cdot}5$ ft., $A = 30{\cdot}03$ sq. ft. ; $V = 18{,}018$ cu. ft.

(*b*) $x = 1{\cdot}0$ ft., $A = 30{\cdot}0$ sq. ft. ; $V = 18{,}000$ cu. ft.

Excess cut, 1200 cu. ft.]

7†. The centre line of a level road, 40 ft. in width, runs due north, and from a point O in the centre line a proposed road is to run N. 30° E. over a plane hill-side, which slopes downwards at its maximum slope of 11° 41′ in a direction due west. The formation width of the proposed road is 30 ft. and the side slopes $1\frac{1}{2}$ to 1, with the gradient rising uniformly from 0 at 1 in 10.

Neglecting camber, calculate the earthwork volume between sections 100 and 1100 ft. from O. (U.L.)

[3304 cu. yd.]

8†. In the following notes which refer to cross-sections along a length of highway, the numerators show the heights above or below formation level of the ground at the centre line and the side slope limits, the denominators giving the corresponding distances from the centre line :

Distance (ft.)	Left	Centre	Right
1600	$\dfrac{-6{\cdot}4}{24{\cdot}6}$	$\dfrac{-14{\cdot}2}{0}$	$\dfrac{-18{\cdot}4}{42{\cdot}6}$
1564	$\dfrac{0}{15{\cdot}0}$	$\dfrac{-4{\cdot}4}{0}$	$\dfrac{-10{\cdot}6}{30{\cdot}9}$
1536	$\dfrac{7{\cdot}5}{22{\cdot}5}$	$\dfrac{2{\cdot}6}{0}$	$\dfrac{0}{15{\cdot}0}$
1500	$\dfrac{15{\cdot}5}{30{\cdot}5}$	$\dfrac{12{\cdot}6}{0}$	$\dfrac{3{\cdot}2}{18{\cdot}2}$

(*a*) Sketch the four cross-sections, also a plan of the length of the road, given that the formation width is 30 ft.

(*b*) Calculate the volume of cutting and filling between 1500 and 1600, using the average end area rule. (U.L.)

[− 17,126 and + 10,873 cu. ft.; but − 16,862 and + 10,916 cu. ft., if without interpolation in central section.]

9. (*a*) Give the definition of a regular prismoid.

Prove that the prismoided formula may be used to calculate, with close accuracy, the volume of a cutting. In your proof, it will be sufficient to give all the necessary steps in your reasoning. It is unnecessary to prove these steps.

(*b*) How is the volume of a cutting on a curve calculated?

(*c*) A 1 per cent up-grade meets a 0·5 per cent down-grade at a chainage of 500 ft., the reduced level of the point of intersection being 100·00 feet. Obtain the reduced levels every 50 feet along a vertical parabola, 400 feet long, inserted between these gradients. Mention the reason why a vertical parabola is a suitable curve to insert and why the formula employed is mathematically correct. (U.B.)

[(*c*) 98·00, 98·45, 98·81, 99·08, *99·25*, 99·46, 99·31, 99·20, 99·00.]

10. What is (*a*) the radius of a 5° circular curve, (*b*) the deflection angle for a subchord of 60 feet on a 5° curve.

A is the area in square yards of the ground covered by a cutting, 800 feet long, and V is the volume of this cutting in cubic yards, on the assumption that the surface of the ground is horizontal at right angles to the longitudinal line of the cutting. If, instead of the ground being horizontal, there had been a surface side slope of 10 to 1, show that the area would have been $A + 1/24\,A$ square yards and the volume would have been $V + 1/24\,V + 200$ cubic yards.

Given: Formation width, $2b = 36$ feet;
side slopes of cutting, 2 to 1;
sections taken at every 100 feet. (U.B.)

11. The surface levels along a line 600 ft. long are as follows:

Chainage in feet	0	100	200	300	400	500	600
Level in feet	69	72	75	80	81	87	94

A cutting is to be made along this line, formation width $2b = 36$ ft.; and side slopes 2 to 1, with the reduced level of formation level 50·00 ft. at chainage 0, the cutting rising uniformly at a slope of 1 in 50.

Calculate the volume of the cutting in cubic yards and the area of the land covered by the cutting in square yards, if the original ground surface has a side slope of 12 to 1 at right angles to the line of the cutting. (U.B.)

[43,470 cu. yd. (trapezoidal); 4,427 sq. yd.]

12. A railway cutting is made in ground having a side slope of 10 horizontal to 1 vertical. The sides of the cutting have a slope of $1\frac{1}{2}$ to 1, and the width of the cutting at formation level is 30 feet. The depth of cut on the centre line is 14 feet. Calculate the area of the cross-section of the cutting and the values of the half-widths. (U.D.)

[734·4 sq. ft., 42·4 ft. and 31·3 ft.]

13. A cutting is to be through ground where the cross slope varies considerably. At A the depth of cut was 10 ft. at the centre line and the cross slope was 10 to 1. At B the corresponding figures were 14 ft. and 12 to 1, and at C 12 ft. and 8 to 1. AB and BC are each 100 feet. The formation width is 30 ft., and the side slopes $1\frac{1}{2}$ to 1. Calculate the volume of excavation between A and C. (I.C.E.)

[4670 cu. yd. (trapezoidal) (4911).]

14. On ground used as a borrow pit, the contours are parallel and, for 5 feet intervals of height, they are 25 feet apart in a horizontal direction. Determine the exact dimensions of a borrow pit, having a level formation, to give 2000 cub. yds. of material, it being assumed that no bulking takes place. The formation of the borrow pit is to be square as seen in plan and the side slopes are to be 1 to 1. (U.G.)

[Square formation 73·92 ft.; width 110·88 ft.; uphill length 92·40 ft.]

ARTICLE 2: EARTHWORK ON CURVES

INTRODUCTION

The effects of curvature are usually considered in final estimates, even though in ordinary cases they are relatively small, and often within the limits of error of measurement and calculation. In simple cases the effect should be estimated before resort is made to correction, whether by direct calculation or by corrections applied to straight volumes. The effect of a curve on the volume can be very considerable in the case of road widenings, and may be appreciable in the case of hill-side sections, where part is in cut and part in fill. The cases considered are those amenable to calculation, complex cases being a matter for graphical methods. In general, estimated centroids lead to appreciable error. The effect on transition curves is even less, being in the ideal case as for a curve of twice the radius of the main curve.

Curved volumes. It now remains to consider the case when the centre line of a cutting or an embankment is on a curve.

In this connection the theorem of Guldinus, or Pappus, is introduced; namely, the volume swept through by a constant area rotating about a fixed axis is equal to the area multiplied by the length of the path traversed by the centroid.

In general, however, the cross-sectional areas will not be constant, but the theorem may be considered satisfactory when the length of the path traced by the centroids is substituted for the length along the centre line. Even so, it would be difficult to find a correction to the lengths l between the sections, since the centroids would be at varying distances

from the centre line for successive sections; and in consequence, a correction is applied to the sectional areas themselves, giving **equivalent areas**, A_e, which embody the eccentricities referred to the centre line of the curve.

Thus if A is the area of a section and e its eccentricity from the centre line, then the volume swept out on a curve of radius R in a short length subtending an angle of θ radians at the centre of curvature, will be approximately $A(R \pm e)\theta$, the *plus* or *minus* sign indicating that the centroid is on the *opposite* or *same* side of the centre line as the centre of curvature. If the eccentricity were neglected, the volume would be $V_0 = AR\theta$, and the error in the length $R\theta$ would be $\pm Ae\theta$, or Ae/R per unit length. Then

$$\text{True vol.} : \text{nominal vol.} :: A(R \pm e)\theta : AR\theta; \text{ or } V = V_0(1 \pm e/R).$$

Hence the equivalent area $A_e = (1 \pm e/R)A = A \pm M/R$, the factor $1 \pm e/R$ being independent of the length $l = R\theta$ in volume calculations.

Also it is immaterial whether the true eccentricity e, or the eccentricity E of the whole area be found; for if M' be the moment of the whole area A', M that of the true area A, and m the moment of the formation triangle, $a = w^2/s$, all about the centre line; then $M' = M + m$; but since $m = 0$ if the formation is assumed to be level across, and $M' = A'E$,

$$M = Ae, \quad \text{or} \quad e = A'E/A.$$

In any case the effect of *cant* (or cross-fall) upon a and m is usually negligible.

Since the distances around curves are usually along chords, not arcs, while the eccentricity is not determinate with great exactness, it is advisable to calculate the volumes derived from the simpler sections by the average end area rule, $V_0 = \frac{1}{2}l(A_1 + A_2)$, which is then modified by the prismoidal correction, and finally by the curvature correction

$$c = \pm \frac{l}{2R}(A_1e_1 + A_2e_2) = \pm \frac{l}{2R}(M_1 + M_2), \quad \text{..............(14)}$$

which is the same whether whole or true areas and their corresponding eccentricities are employed.

On the other hand, it is more expeditious to use the equivalent areas $A(1 \pm e/R)$ directly in the prismoidal rule in determining volumes from complex sections.

Effect of curvature. The effects of curves are often overestimated, particularly in view of the fact that at best the cross-sections only approximate to the lateral configuration of the ground. Thus if curvature is ignored in a series of equal similar sections of true areas A, a distance l apart, the curvature correction c is to the actual curved volume V as

$$\pm \frac{Ael}{R} \text{ is to } A(1 \pm e/R)l,$$

or

$$c/V = \frac{e/R}{1 \pm e/R};$$

and since this ratio will not be affected appreciably if the denominator is written as unity, thus also avoiding the $\pm$ signs,

$$c/V = e/R, \quad \text{.................................(15)}$$

which is conveniently expressed as a percentage value.

Since, however, whole areas A' and their eccentricities E may be used, and $Ae = A'E$, the preceding rule may be written,

$$\frac{A'E}{AR} = \frac{A'E}{(A'-a)R},$$

where a is the area of the formation triangle.

On the other hand, in the case of road widening on one side only, the effect may be considerable, while consideration is always necessary in the case of hill-side sections on curves.

Reference has been made to the small effect of cross-fall or cant when writing the moment m of the formation triangle as zero; and (excepting the case of hill-side sections) this is actually $\frac{\frac{2}{3}w^3}{(1-s^2/n^2)^2}$, where the cross-fall is n horizontally to 1 vertically, uniformly with s and r. Thus with $n = 10$ and $s = 2$, the value of m is $0{\cdot}072w^3$, which is exceedingly small in comparison with the actual moment.

(Incidentally the foregoing rule will follow by similarity with Case (b), p. 106).

Earthwork volumes on transition curves. Normally the effect of curvature is less along spirals than along circular curves, though in the event of improvements introducing transition curves, the volume correction may be considerable, particularly if the widening is on one side of the centre line.

Now the equivalent area $A_e = A(1 \pm e/\rho)$, where ρ is the radius of any point on the spiral, and for the *Clothoid*, $d\lambda/d\phi = \rho = \frac{\alpha}{\lambda} = \frac{RL}{\lambda}$, while for the cubic parabola $y = \frac{x^3}{6RL}$; $\frac{d^2y}{dx^2} = \frac{1}{\rho}$, or $\rho = \frac{RL}{x} = \frac{RL}{\lambda}$, very nearly.

Whence the correction to cross-sectional areas is

$$\frac{Ae}{\rho} = \pm\frac{Ae\lambda}{LR}. \quad \text{.................................(16)}$$

If the areas are similar and equal, the equivalent area $A_e = A\left(1 \pm \frac{e}{2R}\right)$, for on the spiral $\lambda = c$, $2c$, $3c$ up to $Nc = L$, c being the chord length, but in N chords there will be $N+1$ sections affected, the correction being zero at the first section. Hence the average value of λ will be

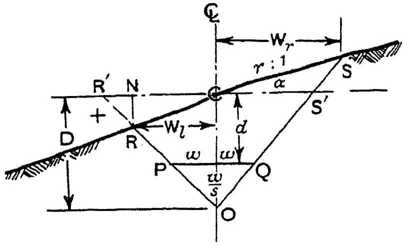


FIG. 54.

In fact, the use of the radius of the hypothetical "one-chord" spiral $\rho = 2R$ affords a useful approximation particularly where there is no great variation in the sections.

Sections on curves. The first four of the following cases may be regarded as standard sections.

(*a*) **Ground level across.** Here no correction is involved, the effect of cross-fall usually being negligible.

(*b*) **Ground sloping transversely** (Fig. 54). For whole areas *left* and *right*,

$$A_l' = \tfrac{1}{2} W_l D \; ; \; A_r' = \tfrac{1}{2} W_r D \; ;$$

and the corresponding moments,

$$M_l' = \tfrac{1}{6} W_l^2 D \; ; \; M_r' = \tfrac{1}{6} W_r^2 D \; ;$$

while the moment of the entire area,

$$A' = -\frac{D}{6}\left\{\frac{W^2}{(1+s/r)^2} - \frac{W^2}{(1-s/r)^2}\right\} = -\frac{2}{3}\,\frac{W^2 D s/r}{(1-s^2/r^2)^2}$$

$$= -\frac{2}{3}\,\frac{(A')^2 s}{rD}\, , \qquad \text{.....................(17)}$$

since $$W_l = \frac{W}{1+s/r}\, ; \quad W_r = \frac{W}{1-s/r} \quad \text{(see p. 94)} \; ;$$

also $$E = M'/A' = -\frac{2}{3}\,A'\,\frac{s}{rD}\, .$$

(*c*) **Hill-side sections** (Fig. 55). For the reasons stated on p. 95, the moments must be treated separately as M_r and M_l, the true eccentricities e_r and e_l being found.

where CN is the distance from the centre line and $SN, S'N'$ are the medians.

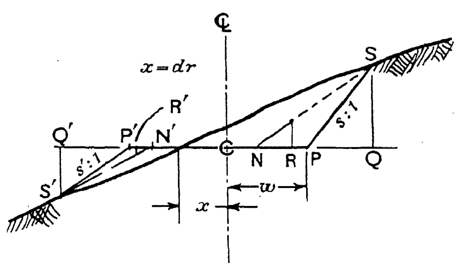


FIG. 55

$$NQ = W_r - CN = W_r - \tfrac{1}{2}(w - x).$$

$$e_r = \tfrac{1}{2}(w - x) + \tfrac{1}{3}(W_r - \tfrac{1}{2}(w - x))$$

$$= \tfrac{1}{3}(W_r + (w - x)) = \tfrac{1}{3}\left\{\frac{w + sd}{1 - s/r} + (w - dr)\right\}.$$

$$e_l = CR' = CN' + \tfrac{1}{3}N'Q';$$

but $\quad CN' = x + \tfrac{1}{2}(w - x) = \tfrac{1}{2}(w + x),$

$$N'Q' = W_l - CN' = W_l - \tfrac{1}{2}(w + x).$$

$$\therefore\ e_l = \tfrac{1}{3}(W_l + (w + x)) = \tfrac{1}{3}\left\{\frac{w - sd}{1 - s/r} + (w + dr)\right\}. \quad \text{......(18), (19)}$$

(*d*) **Three-level sections.** For this case, the most common in field calculations :

$$M_l' = \tfrac{1}{3}W_l \times \tfrac{1}{2}W_l D = \tfrac{1}{6}W_l D;$$

$$M_r' = \tfrac{1}{3}W_r \times \tfrac{1}{2}W_r D = \tfrac{1}{6}W_r D;$$

and the total moment,

$$M' = \tfrac{1}{6}D(W_l^2 - W_r^2);$$

but $\quad A' = \tfrac{1}{2}D(W_l + W_r);$

and $\quad E = \tfrac{1}{3}(W_l - W_r). \quad \text{..................................(20)}$

Example†. A portion of a highway cutting, 400 ft. in length, is on a curve of 500 ft. radius, the original surface of the ground sloping uniformly upwards towards the centre of curvature, 5 horizontally to 1 vertically.

The formation is level and the depth of the centre-line cut is 22·5 ft. throughout the curve.

This portion is to be improved by increasing the formation width from 20 ft. to 30 ft. by excavating on the downhill side only, and retaining the 1 : 1 side slope.

Calculate the volume of additional excavation in cubic yards. (U.L.)

The most expeditious analytical treatment appears to be that of finding the combined centroids of the areas ABC (area a) and $BCFG$ (area hx) (Fig. 56).

$AD = DB \,.\, \tan\alpha = (BC - AD)\tan\alpha$, since $AD = CD/s = CD$ with $s = 1$.

$$AD = \frac{BC\tan\alpha}{1 + \tan\alpha} = \frac{10 \times 1/5}{1\cdot 2} = 1\cdot667 \text{ ft.}$$

Therefore $\bar{z} = 5' + \tfrac{1}{3}ED = 5' + \tfrac{1}{3}(5 - 1\cdot667) = 6\cdot111'$,

$$h = d - HJ, \text{ where } HJ = W_r \tan\alpha = \frac{w + sd}{r(1 + s/r)}$$

$$= d - \frac{w + sd}{r(1 + s/r)} = \frac{d - w/r}{1 + s/r} = \frac{22\cdot5 - 2}{1\cdot2} = 17\cdot08'. \quad \text{..............(1)}$$

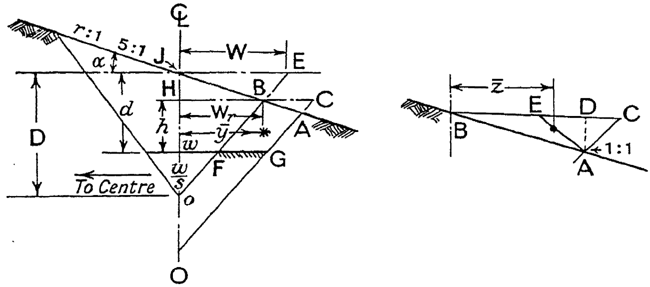


FIG. 56.

$$\bar{y} = w + 5' + \tfrac{1}{2}sh = 10 + 5 + 8{\cdot}54 = 23{\cdot}54'. \quad \ldots\ldots (2)$$

$$\text{(Check: } \bar{y} + \tfrac{1}{2}sh - \tfrac{1}{2}x = W_r = 27{\cdot}08.)$$

$$a = \tfrac{1}{2}BC \times AD = 5 \times 1{\cdot}667 = 8335 \text{ sq. ft.} \quad \ldots\ldots (3)$$

$$W_r = \frac{w + sd}{1 + s/r} = \frac{10 + 22{\cdot}5}{1{\cdot}2} = 27{\cdot}08 \text{ ft.} \quad \ldots\ldots (4)$$

Total moment of area about $C.L. = (hx)\,\bar{y} - a(W_r + \bar{z})$

$= 17{\cdot}08 \times 10 \times 23{\cdot}54 - 8{\cdot}335(27{\cdot}08 + 6{\cdot}111)$

$= 4297{\cdot}27 = Ae$, e being the eccentricity.

Total area, $hx - a = 170{\cdot}8 - 8{\cdot}34 = 162{\cdot}46$ sq. ft.

Equivalent area, $A_e \quad = A(1 + e/R) = A + Ae/R = 162{\cdot}46 + 8{\cdot}59$

$= 171{\cdot}05$ sq. ft.

Additional volume, $A_e l \quad = 400 \times 171{\cdot}05 = 68{,}420$ cu. ft.

$= 2534$ cu. yd.

Example†. In a preliminary earthwork estimate, the volume in cubic yards was calculated by the Average End-area Rule, the effect of a 5° circular curve being ignored. State the error arising from this omission on the following portion of the centre line, given that the formation width is 30 ft., the side slopes $1\frac{1}{2}$ horizontally to 1 vertically, and the lateral slope of the ground $7\frac{1}{2}$ horizontally to 1 vertically, upwards from the centre of curvature.

Station	221	222	223	224	225	226	(100 ft. units)
Depth of cutting	12·4	12·4	13·6	13·8	14·2	14·2 ft.	(U.L.)

Thus, for whole areas A', the area of the formation triangle (w^2/s) having no effect upon the calculations :

Whole areas, level across, $\quad A_0' = sD^2$, where $D = d + w/s$.

„ „ sloping across $\quad A' = \dfrac{sD^2}{1 - s^2/r^2} = \dfrac{1\frac{1}{2}D^2}{0{\cdot}96}$.

Volume correction to end area rule

$$= +\tfrac{1}{2}\frac{l}{R}(A_1'E_1 + 2A_2'E_2 + 2A_3'E_3, \text{ etc.}),$$

where $R = \dfrac{50}{\sin 2\frac{1}{2}^\circ} = 1145$ ft., the following relation existing between the equivalent and whole areas, $A_e = A'\left(1 \pm \dfrac{\omega}{R}\right)$, $A'E$ being the moment of the whole area A', or $\frac{2}{3}(A')^2 \dfrac{s}{rD}$.

Station - - -	221	222	223	224	225	226
Whole depth, D -	22·4	22·4	23·6	23·8	24·2	24·2
Whole area, level -	752	752	838	850	879	879
,, ,, sloping	783	783	874	886	916	916
Moments of latter -	3649	3649	4316	4398	4623	4623

$$\text{Correction} + \frac{100}{2 \times 1145}(3649 + 2(3649) + 2(4316) + 2(4398) + 2(4623) + 4623)$$

$$= +1845 \text{ cu. ft.} = 68{\cdot}32 \text{ cu. yd.}$$

QUESTIONS ON ARTICLE 2

1†. Derive a rule for the equivalent area of a cross-section of a cutting of centre line depth d and formation width $2w$ on a curve of radius R, the side slopes being s horizontally to 1 vertically and the lateral slope of the ground r to 1 likewise. Give a rule for the prismoidal volume of the cutting, embodying the above rule. (U.L.)

2†. Derive a rule for the curvature correction applicable to cross-sectional areas occurring on transition curves of the form $\lambda = m\sqrt{\phi}$, where m is constant and λ and ϕ are the intrinsic co-ordinates with respect to the point of tangency of the spiral and main tangent. Also show that when the areas are equal and similar, a radius of curvature equal to twice the radius of the main curve may be used in the calculations. (U.L.)

3†. Determine the percentage error per 100 ft. due to neglecting a curve of radius R in a cutting of uniform depth d in ground sloping laterally down towards the radius of curvature r horizontally to 1 vertically, the side slopes being likewise s : 1 and the formation width $2w$.

Calculate the error, given that $R = 200$ ft., $2w = 30$ ft., $d = 14{\cdot}8$ ft., $r = 6$, and $s = 1\frac{1}{2}$. [2·6 per cent]

4††. A road around a hill was constructed as follows with its centre line on a circular curve of 400 ft. radius, the ground rising laterally at a uniform slope of 6 horizontally to 1 vertically towards the centre of curvature.

Formation width, 20 ft., level laterally.

Side slopes 2 horizontally to 1 vertically in filling and $1\frac{1}{2}$ to 1 likewise in cutting.

Areas in cutting equal to areas in filling.

Centre line consistently 84·2 ft. above datum.

The road is to be widened to 30 ft. formation by excavation on the hill-side, retaining the $1\frac{1}{2}$ to 1 side slope.

Assuming that the formation is level across, calculate the volume of earth-work for a length of 600 ft. on the original centre line. (U.L.)

[Equiv. area = 38·32 sq. ft., vol. = 22,992 cu. ft. = 851·6 cu. yd.]

5†. A railway cutting in ground which is level transversely is to be widened on one side of the centre-line so that its formation width is increased from 20 ft. to 30 ft., the side slopes remaining $1\frac{1}{2}$ horizontally to 1 vertically.

The portion under consideration is near the first tangent point of a circular curve of 36 chains radius, where the central depth of cutting is 18·4 ft. for 5 chains in either direction, and the widening will occur on the side remote from the centre of curvature. It is, however, proposed to insert transition curves, 6 chains in length, these being either clothoid spirals or cubic parabolas.

Calculate the volume of additional excavation for the length of the first transition curve, assuming the formation to be level across. (U.L.)

[Trapezoidal vol. = 73,306 cu. ft. = 2715 cu. yd.]

6†. The following cross-section notes show the staff readings and corresponding slope stake distances in a series of " three-level sections " which occur on an 8° curve bearing to the *right*.

Station	Left	Centre	Right
75	$\frac{2 \cdot 8}{15 \cdot 7'}$	$\frac{3 \cdot 3}{0}$	$\frac{3 \cdot 6}{14 \cdot 9'}$
74	$\frac{4 \cdot 2}{15 \cdot 0'}$	$\frac{4 \cdot 8}{0}$	$\frac{5 \cdot 0}{14 \cdot 3'}$
73	$\frac{5 \cdot 2}{15 \cdot 4'}$	$\frac{5 \cdot 7}{0}$	$\frac{6 \cdot 2}{14 \cdot 4'}$
72	$\frac{6 \cdot 2}{15 \cdot 1'}$	$\frac{7 \cdot 0}{0}$	$\frac{7 \cdot 5}{13 \cdot 9'}$
71	$\frac{8 \cdot 9}{13 \cdot 7'}$	$\frac{8 \cdot 8}{0}$	$\frac{9 \cdot 2}{13 \cdot 4'}$

Using the average end area rule, calculate the content of the cutting in cubic yards between the 100 ft. stations, Nos. 71 and 75 inclusive, given that the formation width is 20 ft. and the side slopes 1 to 1. (U.L.)

[See " driving slope stakes ". Centre cuts : 5·2, 4·4, 4·9, 4·4, 3·8 ft.]
Curve effect negligible. Vol. 1652 cu. yd.

ARTICLE 3 : MASS-HAUL CURVES

Mass curves are diagrams in which the ordinate represents the total quantity of power, water, or volume, and the abscissa the periods or distances, usually regular, to which the former values are assessed. Thus in the well-known curves of stream flow, or yield, the total quantity in cubic feet per sec., etc., is plotted against the several months of the years of observation. Characteristic of such curves is the fact that the yield is always positive, and so the slope of the mass curve at any point is never negative, the total volume being represented by a horizontal line during the period of drought.

But when the mass method is applied to total volumes V of earthwork, usually in cu. yd., plotted as ordinates against abscissa stations, usually in 100-ft. units, a distinction is made between cuts and fills in that these take the usual convention of plus and minus respectively, and the resulting mass curve consists of positive peaks and negative depressions, points analogous to the maxima and minima in ordinary graphs. It is by virtue of the fact that the slope of a mass curve can thus be positive and negative that the curve can be used to advantage in earthwork estimates.

Before proceeding to the discussion of mass-haul curves, it will be necessary to introduce the various terms and units common to the work.

Haul. Haul is the product of the distance between the centres of gravity of equal volumes of cut and fill and the volume of the cut.

Thus if H is the haul, v the volume of the cut, and d the distance between the centroids of the equalised volumes, $H=vd$. The customary unit of haul is the **station yard**, being the haul involved in moving 1 cu. yd. of earth a distance of 100 ft.

Shrinkage and swell. It will be noted in the foregoing definition that the cut was specified, and this suggests the fact that in the excavation of cuts, the material expands in volume, while after dumping in the fill there is usually some reduction of the increased volume due to consolidation of the mass. Uniform, unmixed material, worked consistently under fixed conditions, may be expected to undergo constant relative changes in excavation, hauling, and dumping. Also uniform mixtures of different materials may be assumed to behave in like manner accordingly. It is, however, beyond the scope of this article to digress upon this subject, the treatment of which will be found in such works as Gillette's *Earthwork and its Cost*, and *Earthwork Haul and Overhaul*, by J. C. L. Fish. The author is particularly indebted to the last work for much information on the subject.

Let v and V represent respectively the volumes in cu. yd. of a given body of the material in place before cutting and in fill after dumping,

Δv the swell-increment, k the swell ratio, q the swell factor and e the balancing or equating factor; then

$$\Delta v = V - v\ ;\quad k = \frac{\Delta v}{v}\ ;\quad q = \frac{V}{v} = 1 + k\ ;\quad \text{and}\quad e = \frac{v}{V} = \frac{1}{q}\,.$$

It is an easy matter to determine the value of the swell factor by cross-sectional measurements during progress of the work in simple cases, but in cases of long stretches of roadbed it is impracticable to measure such cross-sections as would lead to the direct determination of the factor of any single cut or fill, and a series of observations are made and averaged out in order to determine the values applicable to the individual cuts and fills.

Freehaul, overhaul, and crosshaul. In some earthwork contracts no regard is taken of the distance the material is hauled, it being merely stipulated that the price consists in excavating, hauling, and dumping. In others it is stipulated that a certain price shall be paid for excavating and hauling to a specified distance, and that excess payment shall be made when this distance is exceeded. In contracts of the latter class, the specified distance is known as the **freehaul** distance, which is not standard, though 500 ft. appears to be the most popular. The distance in excess of the freehaul is styled the **overhaul** distance, while the term **crosshaul** is applied in cases where the excavated material is carried in both directions over the same part of the line.

Fig. 57 (*b*) shows a mass curve, plotted from cut volumes and balanced fill volumes, and above this, in Fig. 57 (*a*), is the longitudinal section, or

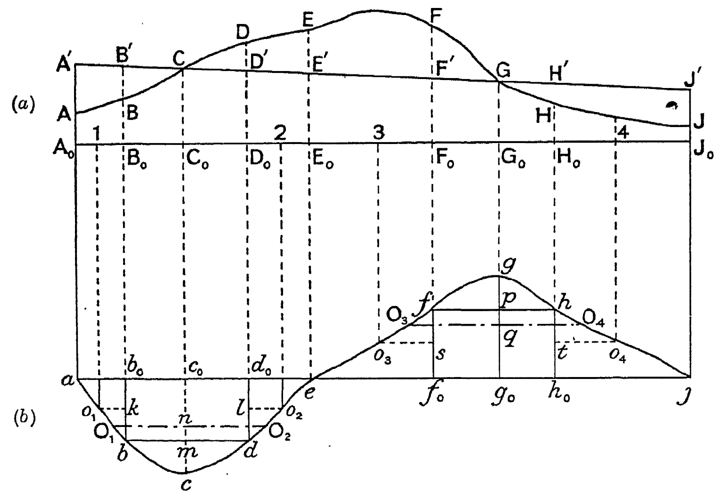


FIG. 57.

profile, for the relevant portion of the roadbed. It will be seen that the ordinates of the lower curve are the *massed* or algebraical sums of the volumes corresponding to the ordinates of the upper curve, and that, in passing from left to right, the **grade points** (of zero cut or fill) occur at the maxima and minima ordinates of the mass curve, the slope of the latter being steepest at stations of greatest volume.

The following characteristics of the curve may be summarised :

(1) The algebraical difference between the ordinates at adjacent maxima and minima points represents the yardage between the two stations.

(2) Any horizontal line drawn to intercept a loop of the mass curve will cut the latter in two points, between which the cut will exactly balance the fill.

Thus if the points b, d be projected vertically to B' and D, the volumes $BB'C$, CDD' will be equal, and the same will apply to $FF'G$ and $GH'H$, as intercepted by the line fh. For this reason bd and fh are termed **balancing lines**, and these will usually represent the limits of freehaul. Such lines are horizontal for curves plotted from cut volumes and balanced fill volumes, and vice versa, thus allowing for swell and shrinkage. Sloping balancing lines indicate that the mass curve is plotted from cut volumes and actual fill volumes. In the bulk of problems, the horizontal balancing line is used, though some surveyors prefer to work with unbalanced volumes and oblique lines accordingly.

(3) If it be assumed that bd and fh are drawn to represent the freehaul, the areas intercepted between these lines and the corresponding loops of the mass curve is a measure of the haul in making the cuts and fills between the extremities of the loops.

Thus, the area cbd represents the haul embodied in making the fill between B_0 and C_0 from the cut between C_0 and D_0, while the extreme haul distance is B_0D_0, balanced volumes of cut and fill being represented by cm in the mass curve.

Then if a is the haul area cbd in square inches, and x and y are the scales, being such that x is the number of 100 ft. stations per inch of abscissa and y the number of cubic yards per inch of ordinate, the haul h in station yards per square inch is xy, and the total haul in the loop bcd is $H = ah$.

Thus if the area A of the loop bcd is 2·6 sq. in. by planimeter, and x and y are respectively 1″ to 200 ft. and 200 cu. yd.,

$$H = 2{\cdot}6 \times 2 \times 200 = 1040 \text{ st. yd.}$$

(4) The centre of mass for a given series of prismoids is determined by bisecting the ordinates of the mass curve, usually at a freehaul limit, and through this point drawing a horizontal line to cut the mass curve.

Thus a horizontal line through k, the midpoint of bb_0, determines o_1, the centre of mass of the volume between A_0 and B_0, the point o_1 being expressed by its projected distance on the base A_0J_0 in 100-ft. stations. Similarly the points l, s, and t lead respectively to o_2, o_3, and o_4 and the projected distances marked 2, 3, and 4 accordingly on the distance line of the longitudinal section.

(5) The average haul distance for a given body of material is the distance between the centre of mass of that body in cut and the corresponding point of that body in fill.

Thus, if the haul represented by the area *abcde* is divided by the ordinate cc_0, the quotient will be the distance represented by O_1O_2 as given by bisecting cc_0 at n, O_1 and O_2 determining respectively the mass centres of the volumes in $AA'C$ and CEE' of the longitudinal section.

(6) The overhaul is represented by the ordinate of a loop between the base and the balancing line.

Thus, the overhaul is the ordinate bb_0 (or dd_0) multiplied by the distance o_1o_2; and, incidentally, $cm+mc_0=cc_0=$ total earthwork in loop *abcde*. Similarly for *efghj*. Had there been no freehaul limit, the haul in the loop *abcde* would be given by the ordinate cc_0 multiplied by the distance O_1O_2.

QUESTIONS ON ARTICLE 3

1†. Describe with neat sketches an "Earthwork Mass Curve", stating the relationship it bears to the corresponding longitudinal section. Show concisely how the "haul" may be ascertained from the curve, also the "overhaul" for an assumed freehaul limit. (U.L.)

2†. The following notes refer to a 1200 ft.-section of a proposed railway; and the earthwork distribution in this section is to be planned without regard to adjoining sections.

The table shows the stations (in 100-ft. units) and the surface levels along the centre line, the formation being at an elevation above datum of 43·5 at Station 70, and thence rising uniformly on a gradient of 1·2 per cent. The corresponding earthwork volumes are recorded in *cub. yds.*, the cut and fill volumes being prefixed respectively with the *plus* and *minus* signs.

(*a*) Plot the longitudinal section, using a horizontal scale of 100 ft. to 1 in. and a vertical scale of 20 ft. to 1 in.

(*b*) Assuming a balancing factor of 0·8 applicable to *fill* volumes, plot the mass-haul curve on a horizontal scale of 100 ft. to 1 in., and a vertical scale of 1 in. to 1000 cub. yds.

(*c*) Calculate the total haul in *station yards* in the section, and indicate the haul limits on the curve and longitudinal section.

(*d*) State which of the following estimates you would recommend: (i) No free haul at 6s. 11d. per cub. yd. for excavating, hauling, and filling. (ii) Freehaul limit of 300 ft. at 5s. 10d. per cub. yd. *plus* 6d. per station yd. for "overhaul", or haul distance exceeding 300 ft.

1 *station yard* = 1 *cub. yd. hauled a unit distance of* 100 *ft.*

Station	Surface level	Volume	Station	Surface level	Volume
70	52·8		76	37·5	
		+ 1860			− 2110
71	57·3		77	41·5	
		+ 1525			− 1120
72	53·4		78	49·5	
		+ 547			− 237
73	47·1		79	54·3	
		− 238			+ 362
74	44·7		80	60·9	
		− 1080			+ 724
75	39·7		81	62·1	
		− 2025			+ 430
76	37·5		82	78·5	

(U.L.)

[(*a*) Cuts/Fills : + 9·3, + 12·6, + 7·5, 0, − 3·6, − 9·8, − 13·2, − 10·4, − 3·6, 0, + 5·4, + 5·4, + 20·6.

(*b*) Reduced vols. : + 1860, + 1525, + 547, − 190, − 864, − 1620, − 1688, − 896, − 190, + 362, + 724, + 2290.

Massed vols. : + 1860, + 3385, + 3932, + 3742, + 2878, + 1258, − 430, − 1326, − 1516, − 1154, − 430, + 1860.

(*c*) Total Haul = 22,443 st. yds. to 81 + 10 ; beyond, carried forward.

(*d*) (i) £1,881 6s. 8d. ; (ii) £2,061 1s. 10d.

3. With reference to civil engineering practice, explain what is meant by the following :

(*a*) the trapezoidal and prismoidal rules,
(*b*) the prismoidal correction,
(*c*) haul, free haul, and overhaul,
(*d*) mass diagram.

4. What is meant by a " mass curve "? Explain fully how it is constructed and what information it gives. (U.D.)

ARTICLE 4 : CIRCULAR CURVES

Definitions. A curve is specified by its radius R, in Gunter chains in the United Kingdom ; in America and elsewhere, by its " degree " D, which is the angle subtended at the centre by a *chord* of 100 ft. The curve is laid down as a chord-angle construction, the chord approximating to the arc within the limits of chaining error. Compound curves are con-

secutive arcs with curvature in the same direction ; to the " right ", or the " left ", as the case may be. Reverse, or serpentine, curves consist of consecutive arcs which bear in opposite directions.

Circular curves were formerly set out by linear methods : (*a*) by **offsets from the tangents,** and (*b*) by **chord offsets**, which are ties of the last chord produced and the corresponding chord of the curve. Occasionally curves are set out by **offsets to the whole chord**, or line between the tangent points. Rankine's method with the theodolite and chain is followed in modern practice, tangential (or deflection) angles being set off from the tangent concurrently with 66 ft. (100 ft.) chords on the curve. Running chainages are usually employed in curve ranging ; that is, distances chained " through " on straights and curves alike. Thus the tangent points occur at uneven distances : 76 miles 3 furlongs 8·24 chains in the United Kingdom, and 176 + 36·2 (100 ft. and 1 ft. units) in America, the length after the " plus " being the **subchord.**

Elements and data. A, B, tangent points (peg and two guard pegs) ...$T.P_1$, $P.T.$, $P.C.$, etc. (Fig. 58).

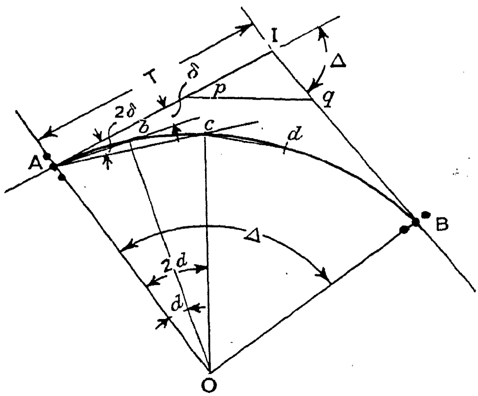


FIG. 58.

Point of intersection, I.

Intersection angle, or (whole) deflection angle, Δ.

Tangent lengths, $T = R \,.\, \tan \tfrac{1}{2}\Delta$.

Tangential (deflection) angles, δ,

$\sin \tfrac{1}{2}\delta = 1/2R$ (by the chord) or

$\delta \quad \dfrac{1719}{R}$ minutes (by the arc) per unit chord, 6_6 ft. or 100 ft.

Degree $D = 2\delta$ for even 100′ chord.

Also $\sin \frac{1}{2}D = \frac{50}{R}$, or $R = \frac{5730}{D}$ ft. approx.

Length of curve, $L = \frac{\Delta}{2\delta}$ in 66′ or 100′ units.

The Rankine method is based upon the proposition that the angle subtended by any arc of a circle at any point on the circumference is one-half the angle subtended by the same arc at the centre. Thus

$$bcA = \tfrac{1}{2}d\,;$$

that is, the tangential angle is one-half the central angle d.

If a and l are the corresponding lengths of the arc and the chord,

$$l = a\left(1 - \frac{a^2}{24R^2}\right),$$

and the error in the chord-arc assumption is

$$\frac{a-l}{a} = \frac{a^2}{24R^2} = \text{ratio of error},$$

which is 1/2000 when l is 1 chain and R, 9·12 chs. Also by the same approximation, $a = \frac{R\,.\,d}{57 \cdot 3}$ with $d = 2\delta$ in degrees, or $\delta = \frac{1719}{R}$ minutes when a is 1 ch.

Curve difficulties. Chief among these are (*a*) **inaccessible intersection points**, and (*b*) **impeded curves**, which latter render it necessary to move the theodolite forward to some point, c, say, on the curve, on account of obstructions or defective sighting.

(*a*) Here points p and q are selected on the tangents, and the interior angles $\alpha + \beta = \Delta$ are measured together with the length pq. Thus by simple trigonometry the lengths Ip, Iq are calculated, and thence the distances to the tangent points A and B (Fig. 58).

(*b*) If a tangent is assumed at a station on the curve, c, say, the angle between this tangent and the chord cA will be 2δ, and from this tangent to the forward station d the angle will be δ, making a total of 3δ from the forward prolongation of Ac. In general, the procedure is as follows with the instrument at any intermediate station m and a back sight on any rear station n. Sight the rear station n with the plates clamped at the tangential angle recorded for this station for a theodolite at A, transit the telescope, and set off the angle, likewise recorded for the station ahead of the theodolite, proceeding as though the theodolite were at the first tangent point A.

Transitting should be avoided unless the instrument is in perfect adjustment by setting the vernier at 180° plus or minus the angle recorded for n.

Compound and reverse curves. Compound curves are formed by the succession of two curves of different radii, R and r, the tangent lengths of which are accordingly T and t, corresponding to central angles α and β, where $\alpha+\beta=\Delta$ (Fig. 59).

The following general formulae may be derived from the trigonometrical relations:

$$\text{vers}\,\alpha=\frac{t\sin\Delta-r\,\text{vers}\,\Delta}{R-r};\quad \text{vers}\,\beta=\frac{R\,\text{vers}\,\Delta-T\sin\Delta}{R-r}.$$

Reverse curves are used when the straights are parallel or include a very small angle of intersection. Whenever practicable, they are avoided by the insertion of a short straight or a reversed transition curve. The elements are not directly determinate unless some condition or dimension is specified, as for example, equal radii ($R=r$) or equal central angles ($\alpha=\beta$), the general case being that $\Delta=\pm(\alpha-\beta)$, according as the point of intersection occurs before or after the reverse curve. The general equation is:

$$\cos(\alpha-x)=\frac{R\cos x+r\cos y}{R+r}=\cos(\beta-y)$$

where x and y are the angles at the tangent points between the tangents and the line between the tangent points (Fig. 60).

Example†. The following notes refer to the setting out of a circular curve to the right of 15 chains radius between two straights AB, BC, the intersection B of which was inaccessible:

Measurement $ab=6{\cdot}21$ chs. from a in AB to b in BC.

Theodolite at a: interior angle $\alpha=23°\ 43'$.

,, ,, b: ,, ,, $\beta=25°\ 54'$.

Chainage of $a=29{\cdot}059$.

Owing to obstructions it will be impossible to set out angles from the tangent at the first tangent point A beyond that to the peg at 31·0 chains on the curve, and the theodolite will be set up at this peg in order to continue the curve to the second tangent point B.

Describe concisely the procedure of setting out from an intermediate peg on the curve, and show in tabular form the tangential angles to be set out at A and at 31·0 chs. for pegs at even chains on the curve, giving also the nearest readings for a vernier reading to 20″. (U.L.)

$$\Delta=\alpha+\beta=49°\ 37';\quad aB=\frac{ab\sin\beta}{\sin\Delta}=\frac{6{\cdot}21\times0{\cdot}43680}{0{\cdot}76173}=3{\cdot}561\text{ chs.}$$

Tangent lengths: $R\tan\frac{1}{2}\Delta=15\times0{\cdot}46224=6{\cdot}934$ chs.

Chainage of $A = 29{\cdot}059 + 3{\cdot}561 - 6{\cdot}934 = 25{\cdot}686$. (1st T.P.)

$$\text{Tangential angle per chain} = \frac{1718{\cdot}9}{R}\ \text{mins.} = 1^\circ\ 54{\cdot}59'$$

$$\text{Length of curve} = \frac{\Delta}{2\delta} = \frac{2977}{229{\cdot}18} = 12{\cdot}989\ \text{chs.}$$

Chainage of $B = 25{\cdot}686 + 12{\cdot}989 = 38{\cdot}675$ chs. (2nd T.P.)

Angle for 1st subchord : $0{\cdot}314 \times 114{\cdot}59' = 35{\cdot}98'$.

„ „ last „ $0{\cdot}675 \times 114{\cdot}59' = 1^\circ\ 17{\cdot}38'$.

The tabular form is made up as though the instrument would be stationed wholly at A ; and the tangential angles are calculated preferably by adding successive values of $1^\circ\ 54{\cdot}59'$ to $35{\cdot}98'$, the angle for the first subchord, while the vernier angles are these values taken to the nearest $20''$.

(i) **If instrument is in perfect adjustment.** Backsight F.R. with A vernier at tabular value for peg thus sighted ; transit F.L., and proceed as if theodolite were at A.

(ii) **If adjustment is not perfect or is suspected.** Backsight F.L. with A vernier at tabular value for peg sighted $+180^\circ$; turn in azimuth and proceed F.L. as if at A. This is better than using B vernier at 180° plus tabular angle in backsight and using A vernier for foresight, as the effect of eccentricity will be introduced.

In the problem, the A vernier will be set at 0°, since the backsight will be taken on A.

Example†. The following data refer to a compound circular curve which bears to the right :

Angle of intersection (or total deflection), 59° 45′.
Degree of 1st curve, 3° ; degree of 2nd curve, $4\frac{1}{2}^\circ$.
Point of intersection at 164 + 25 (100 ft. units).

Determine in 100 ft. units the running distances of the tangent points and the point of compound curvature, given that the latter point is 4 + 26 from the point of intersection at a back angle of 294° 32′ from the first tangent. (U.L.)

$$R = \frac{50}{\sin\frac{1}{2}D} = \frac{50}{\sin 1\frac{1}{2}^\circ} = 1909{\cdot}85\ \text{ft.} \quad r = \frac{50}{\sin 2\frac{1}{4}^\circ} = 1273{\cdot}56\ \text{ft.}$$

$$\sin\tfrac{1}{2}\alpha = \frac{IC \,.\, \sin 65^\circ\ 28'}{AC} = \frac{426 \sin 65^\circ\ 28'}{2R\sin\frac{1}{2}\alpha}$$

$$= \sqrt{\frac{426 \sin 65^\circ\ 28'}{2 \times 1909{\cdot}85}} = \sqrt{0{\cdot}10146} = 0{\cdot}31853.$$

$\alpha = 2\,(18°\,34'\,27'') = 37°\,8'\,54''$; $\beta = 22°\,36'\,6''$.

$$T = \frac{R \text{ vers } \Delta - (R - r) \text{ vers } \beta}{\sin \Delta} \quad \frac{1909{\cdot}85 \times 0{\cdot}49623 - 636{\cdot}29 \times 0{\cdot}07680}{0{\cdot}86384}$$

$= 1040{\cdot}54$ ft.

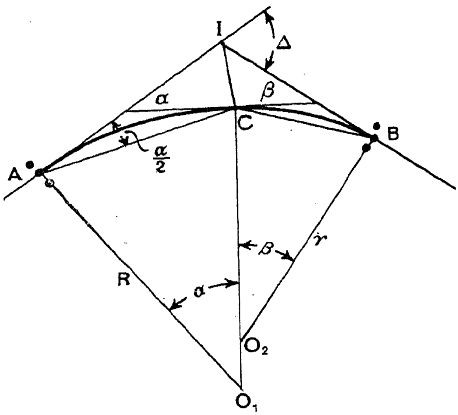


Fig. 59.

Curve length $L = \dfrac{\alpha}{2\delta_1} = 1238{\cdot}28$ ft.

,, ,, $l = \dfrac{\beta}{2\delta_2} = 502{\cdot}26$ ft.

Chainage I	$= 164 + 25$
T	$= 10 + 40{\cdot}54$
Chainage A	$= 153 + 84{\cdot}46$
Length L	$= 12 + 38{\cdot}28$
Chainage C	$= 166 + 22{\cdot}74$
Length l	$= 5 + 02{\cdot}26$
Chainage B	$= 171 + 25{\cdot}00$

Compound Curves. Incidentally, the following problems may arise in connection with compound circular curves :

(i) Given R, r, t, and Δ, to find T.
(ii) Given R, r, T, and Δ, to find t.
(iii) Given R, r, Δ, and L or l, to find T and t.
(iv) Given T, R, and Δ, to find t and r, the point of compound curvature C being also given.

Cases (i) and (ii) introduce the versine formulae (p. 118) ; Case (iii) the additional relations $\alpha = 2L \sin^{-1} \dfrac{1}{2R}$, or $\beta = 2l \sin^{-1} \dfrac{1}{2R}$; and (iv) $\alpha = 2 \sin^{-1} \dfrac{AC}{2R}$, the inverse sine terms being the tangential (or deflection) angles, L and l the lengths of the relevant circular arcs, and β generally $(\Delta - \alpha)$ (Fig. 59).

Example†. AB and CD are two straights such that A and D are on opposite sides of a common tangent BC ; and it is required to connect AB and CD with a reverse curve of radius R.

Given that the angles ABC and BCD are respectively 148° 40′ and 139° 20′ and that BC is 16·28 chs., determine the common radius R and the chainage of the points of tangency and reverse curvature, the direction being from A to D and the chainage of B 145·20 chains. (U.L.)

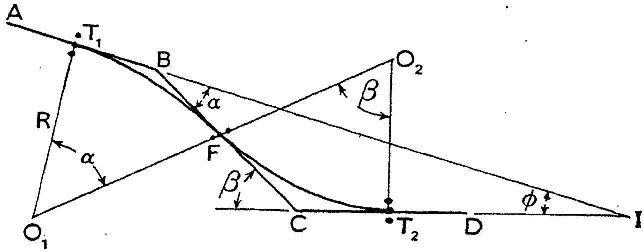


FIG. 60.

$BF = R\tan\frac{1}{2}\alpha$; $FC = R\tan\frac{1}{2}\beta$; $BC = R(\tan\frac{1}{2}\alpha + \tan\frac{1}{2}\beta)$.

$$R = \frac{16{\cdot}28}{\tan 15^\circ 40' + \tan 20^\circ 20'} = \frac{16{\cdot}28}{0{\cdot}28046 + 0{\cdot}37057} = 25{\cdot}00 \text{ chs.}$$

1st tangent length $= R\tan\frac{1}{2}\alpha = 25(0{\cdot}28046) = 7{\cdot}012$ chs.

Tangential angle per ch., $\delta = 1719/R = 1719/25 = 68{\cdot}76$ mins.

Length of 1st arc $= \frac{\alpha}{2\delta} = \frac{1880}{2\times 68{\cdot}76} = 13{\cdot}671$ chs.

,, 2nd ,, $= \frac{\quad}{2\delta} = \frac{2440}{2\times 68{\cdot}76} = 17{\cdot}743$,,

Chainages : 1st tangent point $= 145{\cdot}20 - 7{\cdot}012 = 138{\cdot}188$.
Point reverse curvature $= 138{\cdot}188 + 13{\cdot}671 = 151{\cdot}859$.
2nd tangent point $= 151{\cdot}859 + 17{\cdot}743 = 169{\cdot}602$.

QUESTIONS ON ARTICLE 4

1†. Two adjacent straights have bearings of N. 18° 46′ E. and N. 68′ 16′ E., and these are to be connected by a curve of 40 chs. radius with 1 chain chords and tangential angles.

Prepare a suitable location note form and insert the chainages of the tangent points and the middle point of the curve, the chainage of the point of intersection being 126·54. (U.L.)

[108·10 ; 125·368 ; 142·635]

2*. Two straight lines PQ and QR on the centre-line of a proposed road on a rocky headland are to be connected by a circular curve of 600 ft. radius. From the traverse notes it is found that if the bearing of PQ is assumed to be N. 0° 0′ E., the bearing of QR will be N. 48° 20′ E., while if P be taken as the origin of co-ordinates, the latitude and departure of R will be + 725 ft. and + 365 ft. respectively.

Determine the distances of the tangent points of the curve from the stations P and R. (U.L.)

[$P \ldots T_1 = 134{\cdot}76$ ft. ; $R \ldots T_2 = 216{\cdot}91$ ft., $P \ldots Q$ being 403·96 ft.]

3*. In setting out railway curves it is usual to assume that the lengths of the chords and circular arcs are equal, the error in linear measurement in this class of work not exceeding 1 in 2000.

Determine on this basis the limiting radius or degree for which respective chords of 1 chain and 100 ft. may be used. (U.L.)

[9·13 chs. ; about $6\frac{1}{4}°$]

4††. Two railway straights T_1AI and IBT_2 meeting in an inaccessible point I are to be connected by a compound circular curve such that the arc T_1C of radius 30 chains is equal in length to the arc CT_2 of radius 20 chains, C being the point of compound curvature. You are given the following data :

Line	W.C. bearing	
T_1AI	55° 30′	Chainage of A, 154·23 chains
IBT_2	114° 45′	$AB = 12{\cdot}63$ chains
AB	82° 36′	

(*a*) Prepare a sketch giving all the distances necessary for pegging T_1, C, and T_2 initially.

(*b*) Submit in tabular form complete notes for setting out the curve by tangential angles, pegging through chainages. (U.L.)

[IA, 7·823 chs., IB, 6·697 chs. ; $\alpha+\beta = 23° \, 42' + 35° \, 33' = \Delta = 59° \, 15'$; Lengths L, l, 12·408 chs. Tangent lengths : $T = 14{\cdot}891$ chs. ; $t = 12{\cdot}355$ chs.

(*a*) Measure 7·068 chs. from A to T_1 and 5·658 chs. from B to T_2.

(*b*) Chainages : C, 166·638 ; T_2, 179·046.

Tangential Angles from tangent at T_1, 0·77(57·3′) + 57·3′ + 57·3′ ; etc.

,, ,, ,, ,, C, 0·362(85·9′) + 85·9′ + 85·9′ ; etc.]

5. Discuss the relative merits of designating railway curves by radius and by degree.

AB and BC are successive railway straights, the bearings AB and BC being 128° 23′ and 144° 35′ respectively. They are to be connected by a circular railway curve, which is required to have its midpoint at about 40 feet from B. Calculate to the nearest $\frac{1}{4}°$ the degree of the curve that will suit. Obtain the tangent lengths and, assuming that the chainage at the commencement of the curve is found to be 23,040 feet, compute that at the end. (U.D.)

[$1\frac{1}{2}°$; $T = 539{\cdot}56$ ft. ; 24,120 ft.]

6. C is the inaccessible intersection point of two railway straights AC and CB, which are to be joined by a circular curve. Points D and E are selected on AC and CB respectively, and the following bearings are observed : AD, 173° 42′ ; DE, 187° 24′ ; and EB, 198° 18′. The distance DE is measured, and found to be 960 feet. If on examination of the ground it is found desirable to locate the initial tangent point of the curve at D, obtain the necessary radius in feet and the distance from E to the second tangent point. (U.D.)

[$R = 2{,}000$ ft. ; 110·09 ft.]

7. A reverse curve AB is to be set out between two parallel railway tangents 50 ft. apart. If the two arcs of the curve are to have the same radius, and the distance between the tangent points A and B is to be 400 ft., calculate the radius.

The curve is to be set out by means of offsets from AB at 25 ft. intervals along that line. Calculate the lengths of the offsets. (I.C.E.)

[$R=800$ ft.; for each branch, 0, 2·76, 3·26, 5·90; 6·28; 5·90, 3·26, 2·76, 0 ft.]

8. Readings taken by a theodolite set up at the intersection point A of two straights to points B, D and E are as follows:

Point B - -	$0° \; 0' \; 0''$
Point D - -	$30° \; 0' \; 0''$
Point E - -	$150° \; 0' \; 0''$

The measured length of AD is 60 feet.

Obtain the radius of a circular curve which will connect the straights BA and EA and which will pass through the point D. The running chainage of A, measured along straight BA is $700+61$ feet. Find the chainage of the points B, D, and E, and prepare the table for setting out part B to D of the curve. (U.G.)

[$R=1181·2$ ft.; chainages: B, $697+44·5$; D, $700+11·4$; E, $703+62·9$; tangential angles: $1° \; 20·6'$, $3° \; 46·1'$; $6° \; 11·6'$; $6° \; 28·3'$.]

9. AD is a circular arc which is to be set out by deflection angles round the shoulder of a hill. From the initial tangent point A the part AB, about one-third of the whole curve, can be set out. By transferring the theodolite to B, the setting out may be continued for a similar distance to C, from which it can be completed. Describe in detail how you would manipulate the theodolite at B and C. The method should preferably be such that a list of the deflection angles referred to the tangent at A may be utilised throughout. (I.C.E.)

10. A sharp curve occurs in a mountain-pass project, involving a deviation angle of 150°. Describe how to set out a suitable curve for a small speed of vehicles. It may be assumed that the theodolite has to be shifted to two intermediate positions in each half of the curve in the course of the setting out. It is also assumed that the intersection point is inaccessible, but that the middle point on the top of the curve can be located with reference to the tangents by ordinary survey methods. The radius of curvature may be taken as 100 feet. (T.C.C.E.)

11. Having obtained complete data for setting out a simple circular curve, the laying out work is commenced in the field. After turning seven deflection angles it is discovered that further angles cannot be turned from the existing position of the instrument owing to intervening obstacle to line of sight. Describe briefly the method you would employ in setting out the remaining portion of the curve.

How would you peg out the centre line of a railway by using the offset method, when a peg has to be put in at each chain length? Show how the offsets are calculated. (T.C.C.E.)

12. What do you understand by the following forms of curves and where are they generally used?

1. Lemniscate.
2. Compound curve.
3. Reverse curve.

If in a compound curve the directions of two straights and one radius are known, how will you find out analytically the value of the other radius? (T.C.C.E.)

ARTICLE 5: TRANSITION CURVES

Curves of adjustment, or **easement,** are introduced as **approaches** from the tangent to the circular portion of a railway curve in order (1) to afford a gradual increase of curvature from zero at the tangent point to the specified radius or degree of the circular arc, and (2) to afford the elevation of the outer rail in accordance with such curvature so that the full **superelevation** is attained simultaneously with the curvature of the circular arc.

Superelevation and **cant,** though used synonymously, are respectively the amount by which the outer rail is raised above the transverse level, and the transverse slope accordingly. The relation between the radial and vertical accelerations of the mass of the moving train are embodied in the following relation:

$$e=\frac{GV^2}{990R},$$

where e is the superelevation in ft., V the speed in m.p.h., and R the radius in chs., G the distance between the rail centres being 5 ft. approximately for the standard gauge of $4'\ 8\frac{1}{2}''$.

In 1860 Froude introduced the cubic parabola, the curves being set out, as they still often are, by offsets from the tangent. Multiform compound curves were also used with uniformly increasing curvature, the construction approximating to the cubic effectively. These polychord spirals were popular in America, where Searle's "Railroad Spiral" was used extensively. In 1900 Glover suggested the use of Bernoulli's **clothoid**, and the first approximation to this is a cubic parabola, the relations augmenting the data of the latter curve.* The exact clothoid is also used, and, more recently, the **lemniscate**, particularly for highway transitions.

* *The Transition Spiral.* A. L. Higgins. (Constable & Co.)

In new work the main circular arc is shifted inwards, in order to admit the spirals ; and in amending old track, the main curve is either sharpened, or sharpened and shifted, in order to give the necessary **shift.** The cubic

$$y = x^3/6RL$$

is set out by rectangular offsets y, (a) at distances x from the spiral tangent point on the tangent, or (b) half in this way and half from the redundant piece of the circular arc. The clothoid

$$\lambda = m\sqrt{\phi} = \sqrt{2RL\phi},$$

and tabular spirals are normally set out with deflections from the tangent at the tangent points in conjunction with chords measured around the spiral, the angles of the cubic varying as the squares of the distances thus measured from the tangent point.

The spiral may be set out in various ways : for example, (1) with $\frac{1}{2}$ ch. chords, chained through, with deflection angles $\omega = 573/RL$ mins. per (ch.)^2 ; (2) N chords, 10 say, and $\omega = n^2k$, where $k = 573L/n^2R$, n being the number of the chord counted from the tangent point ; (3) with constant unit deflection, $k = 1'$ (or $\frac{1}{2}'$) so that the chord $c = \frac{1}{24}\sqrt{RL}$, very nearly, c, R and L being in chain units.

The lengths of transition L were formerly calculated as **uniform approaches,** $L = bV^2/R$, being 300 or 500 (or even 1000) times the maximum superelevation e. **Time approaches** have been used in recent times, the rate of gain of superelevation being constant, so that $L = aV^3/R$. Shortt's rule is in this category, and is based upon a limiting rate of gain of radial acceleration of 1 ft. per (sec.)^3. This leads to $L = V^3/1381R$ (chain units), which reduces to $L = \sqrt{R}$ when the main curve carries the maximum speed for its radius R chs., namely, $11\sqrt{R}$, or in the compromise $11{\cdot}19\sqrt{R}$ m.p.h. Comparable with this is the American rule, $L = 600/D°$ ft. Shortt's rule is also used in connection with highway transitions, failing a better expedient.

Data and elements. *Shift* $s = L^2/24R$ (Fig. 63).

Total tangent length $IP = (R+s)\tan\frac{1}{2}\Delta + \frac{1}{2}L\left(1 - \frac{1}{5}\frac{s}{R}\right)$.*

Total deflection angle $\Omega = 573\ L/R$ mins. $= \frac{1}{3}\Phi$, where Φ is the total spiral angle.

Deflection angles $\omega = 573\dfrac{n^2c^2}{RL}$ mins. generally, where c is the chord length and n the number of the chord.

Total abscissa $X = L\left(1 - \frac{3}{5}\frac{s}{R}\right)$.

* *The Transition Spiral.* A. L. Higgins. (Constable & Co.)

Length of circular portion = $\Delta - 2\Phi$, where Δ is the external angle of intersection and δ the tangential angle per chain.

The dimensions X and PA' are very nearly L and $\frac{1}{2}L$ when R is greater than 25 chains, following the first approximation of p. 128. When, as in highway curves, the radius is comparatively small, a higher approximation must be used, rendering the calculations more complex (Fig. 62).

Characteristic of transition curves are the facts that the shift bisects the transition curve and the transition curve the shift ; while the terminal length of transition is equal to the portion of circular arc it replaces.

Cubic parabola. Let G be the distance between the rail centres, v the speed in f.p.s., and e the superelevation in feet corresponding to the radius of curvature ρ at any point on the transition.

$$e = \frac{Gv^2}{g\rho}.$$

Let $1/k$ be the gradient of approach, so that the corresponding distance along the tangent from P is

$$x = ke, \text{ and } x = kGv^2/g\rho, \text{ or } \rho = kGv^2/gx. \quad\ldots\ldots\ldots\ldots\text{(i)}$$

But the curvature $$\frac{d^2y}{dx^2} = \frac{1}{\rho} = \frac{gx}{kGv^2}. \quad\ldots\ldots\ldots\ldots\text{(ii)}$$

Integrating, $dy/dx = gx^2/2kGv^2$ ...... (when $x = 0$, $dy/dx = 0$) ; ......(iii)

likewise $y = gx^3/6kGv^2$ .... ($x = 0$, $y = 0$). ....(1)

Also when $x = L$, $\rho = R$, and from (i), $LR = kGv^2/g$,

and upon substituting this value in (1),

$$y = \frac{x^3}{6RL}. \quad \ldots\ldots \quad (2)$$

The transition curve may be set out either (a) wholly to $x = L$ with rectangular offsets y from the tangent with the terminal ordinate $QN = 4s$, or (b) half from the tangent and half from the redundant portion of the circular curve, the maximum ordinate of $\frac{1}{2}s$ occurring at the centre (Fig. 61).

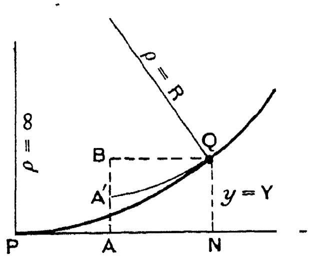


FIG. 61.

Shift. Consider Fig. 61, where PQ is the transition curve and $A'Q$ the redundant portion of the circle of radius R, ρ being the radius of curvature at any point and s the shift.

Now if PQ is the cubic parabola $y = \dfrac{x^3}{6RL}$;

then when $x = L$, $y = Y$, and $Y = AB = QN = \dfrac{L^2}{6R}$.

But by the circle, $A'B = \dfrac{L^2}{8R}$; whence $s = AA' = \dfrac{L^2}{6R} - \dfrac{L^2}{8R} = \dfrac{L^2}{24R}$.

This is also regarded as the basic value of s for the following curve.

The clothoid. Assume the equation

$$\lambda = f(\phi), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where λ and ϕ are the intrinsic co-ordinates of any point on the spiral with respect to an origin at the point of spiral P (Fig. 62).

Now a fundamental requirement of a transition curve is that the radius of curvature ρ shall vary inversely as the length of transition, or

$$\rho = \alpha/\lambda. \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

But
$$\rho = d\lambda/d\phi = \frac{\alpha}{\lambda}, \quad \ldots\ldots\text{(iii)}$$

and upon integrating,

$$\lambda^2 = 2\alpha\phi, \text{ or } \lambda = \sqrt{2\alpha\phi}, \quad \ldots\ldots\text{(iv)}$$

while if the constant m^2 replaces the constant 2α,

$$\lambda = m\sqrt{\phi}. \quad \ldots\ldots\ldots\ldots\text{(1)}$$

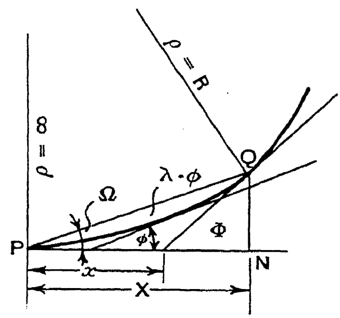


FIG. 62.

Also the superelevation e at any point equals av^2/ρ, which for a constant speed becomes β/ρ : hence

$$e \propto 1/\rho = \frac{\lambda}{\alpha} = k\lambda. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

Thus if e and λ are the running co-ordinates of the gradient of approach, that gradient is the linear relation $e = k\lambda$, which also shows that ϕ^n can have no other index than $n = \frac{1}{2}$, as in Eq. iv.

When the spiral meets the circle at Q, $\lambda = L$ and $\rho = R$ in Eq. iii, so that

$$\alpha = LR \text{ and (iv) becomes } \lambda = \sqrt{2RL\phi}. \quad \ldots\ldots\ldots\ldots\ldots\text{(2)}$$

Now in Cartesian co-ordinates,

$$dx = d\lambda \cos\phi = \frac{m\cos\phi}{2\sqrt{\phi}}, \quad \ldots\ldots \quad \text{(vi)}$$

$$dy = d\lambda \sin\phi = \frac{m\sin\phi}{2\sqrt{\phi}}; \quad \ldots\ldots \quad \text{(vii)}$$

and upon expanding $\cos\phi$ and $\sin\phi$, and integrating dx and dy accordingly,

$$x=m\sqrt{\phi}\left(1-\frac{\phi^2}{5\,.\,2!}\cdots\right); \qquad \text{(viii)}$$

$$y=m\sqrt{\phi}\left(\frac{\phi}{3}-\frac{\phi^3}{7\,.\,3!}\cdots\right). \qquad \text{(ix)}$$

Whence the tangent of the angle of deflection ω from the tangent at P is to the first approximation,

$$y/x=\phi/3;$$

or

$$\omega=\phi/3; \qquad \text{(x)}$$

leading to the cubic, which actually represents most spirals in practice.

Thus $$x=m\sqrt{\phi} \text{ and } y=\frac{m}{3}\phi^{3/2},$$

and, on substituting for ϕ in terms of x, $y=x^3/3m^2$; but since $m^2=2RL$,

$$y=x^3/6RL. \qquad \text{(3)}$$

Finally, it follows from (x) and (2) that $\lambda^2=6RL\omega$, whence

$$\omega=\lambda^2/6RL \text{ radians}=573\lambda^2/RL \text{ minutes}; \qquad \text{(4)}$$

or, if there are n chords each of length c in the length λ,

$$\omega=n^2k, \qquad \text{(5)}$$

where k is the unit deflection, $573c^2/RL$. ......(6)

There are three practical methods of setting out the spiral with the theodolite: (*a*) constant unit deflection (1′, say) by putting $k=1$ in (6), leading to the approximation $c=\frac{1}{24}\sqrt{RL}$; (*b*) constant chord length (33 ft., say) with $\omega=573/RL$ mins. per (chain)^2 or (100 ft.)^2, a process amenable to "chaining through" on spiral and curve alike; (*c*) constant number of chords ($N=10$, say) with $Nc=L$ in (6), or $k=573L/N^2R$ mins.

The foregoing approximations are not applicable when R is small and in certain cases of highway spirals, the more exact use of the subsidiary equations (viii) and (ix) being involved (see p. 136).

Example†. The bearings of two intersecting straights on the centre line of a proposed railway are respectively N. 26° 24′ E. and N. 64° 18′ E., the point of intersection occurring at 56·40 Gunter chains. A circular curve of 36 chs. radius and suitable transition curves are to be inserted, the latter being $\sqrt{R}$ chs. in length, on the assumption that the circular curve will carry the maximum speed for its radius R. Submit a tabular form suitable for the notes of the composite curve, and insert the data relative to the junctions of the transition curves with the straights and the circular arc. (U.L.)

Simple tangent length,

$$Ia = Ia' = R \tan \tfrac{1}{2}\Delta = 36\,(\tan 18^\circ\ 57') = 12{\cdot}361 \text{ chs.}$$

Shift $s = L^2/24R = 0{\cdot}042$ ch., L being length of transition in chains.
Shift increment $aA' = s \tan \tfrac{1}{2}\Delta = 0{\cdot}042 \times 0{\cdot}343351 = 0{\cdot}0095$ ch.
Spiral extension $A'P = \tfrac{1}{2}L(1 - \tfrac{1}{5}s/R) = \tfrac{1}{2}L = 3$ chs. when $R > 25$ chs.
Total tangent length $Ia + aA' + A'P = 15{\cdot}37$ chs.
Total spiral angle $\Phi = L/2R$ rads. $= 3 \times 573L/R$ mins. $= 4^\circ\ 46{\cdot}5'$.
Total deflection angle $\Omega = 1/3\,\Phi = 1^\circ\ 35{\cdot}5'$.

Tangential angle δ per chain of circular arc $= \sin^{-1} \dfrac{1\cdot}{R}$; $= 47{\cdot}74'$; or by approximation, $\dfrac{1719}{R} = 47{\cdot}75'$.

Length of circular arc $QQ' = \dfrac{\Delta - 2\Phi}{2\delta} = \dfrac{18^\circ\ 57' - 9^\circ\ 33'}{95{\cdot}48'} = 5{\cdot}907$ chs.

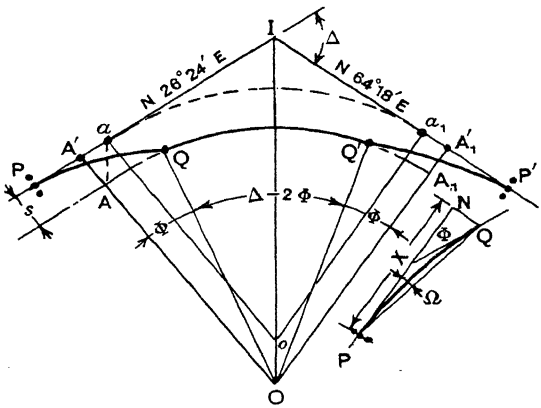


FIG. 63.

Chainage of P $= 56{\cdot}40 \ - 15{\cdot}37 \ = 41{\cdot}03$.
" " $Q = 41{\cdot}03 + 6 = 47{\cdot}03$.
" " $Q' = 47{\cdot}03 + 5{\cdot}907 = 52{\cdot}937$.
" " $P' = 52{\cdot}937 + 6 = 58{,}937$

Q and Q' might be inserted by the total deflection Ω and the long chord PQ (PQ'), where $PQ = X \sec \Omega = 6$ chs. very nearly, X, the total abscissa being $L(1 - \tfrac{3}{5}s/R) = 5{\cdot}996$ chs.

The data may be tabulated as over :

Station	Bearing	Tang. and deflection angles	Vernier	Notes
41 41·03	N. 26° 24′ E.		Reset 00° 00′	 T.P$_1$(P) 1st 6 ch. spiral
47·03			1° 35½′	P.S.C. (Q) 36 chs. rad. curve
52·937			4° 42′	P.C.S. (Q′) from tang. at P.S.C.
58·937 59	 N. 64° 18′ E.		3° 11′	T.P$_2$ (P′) by 2 Ω from tang. at P.C.S.

Example†. On a proposed railway two straights intersect at chainage 12 miles, 77·846 chs. with a left deflection of 37° 44′. It is proposed to put in a circular arc of 16 (Gunter) chains radius with transition curves 4 chains long at each end.

Make the necessary calculations for setting out pegs at 12 m. 76 chs. and 13 m. 3 chs. and the two contact points of the three curves. Describe how you would set them out in the field and establish the direction of the tangent to the circular arc. (U.L.)

Shift $s = \frac{1}{24}$ ch. $= 0{\cdot}042$ ch., since $L = \sqrt{R}$.

Total tangent length $= (R + s) \tan \frac{1}{2}\varDelta + \frac{1}{2}L$ nearly

$= (16{\cdot}042) \tan 18° \, 52' + 2 = 7{\cdot}482$ chs.

Total spiral angle $= L/2R$ rad.

$= 3 \times 573 L/R$ mins. $= 7° \, 9' \, 45''$.

Total deflection angle Ω $= \frac{1}{3}\varPhi = 2° \, 23' \, 15''$.

Central angle of circular arc $= \varDelta - 2\varPhi$ (Fig. 64)

$= 37° - 14° \, 19' \, 30'' = 23° \, 24' \, 30''$

Length of circular portion $= \dfrac{\varDelta - 2\varPhi}{2\delta}$,

where $$\delta = \frac{1719}{R} \text{ mins.} = 107{\cdot}44' ;$$

giving $$L_0 = \frac{1404{\cdot}5}{107{\cdot}44 \times 2} = 6{\cdot}536 \text{ chs.}$$

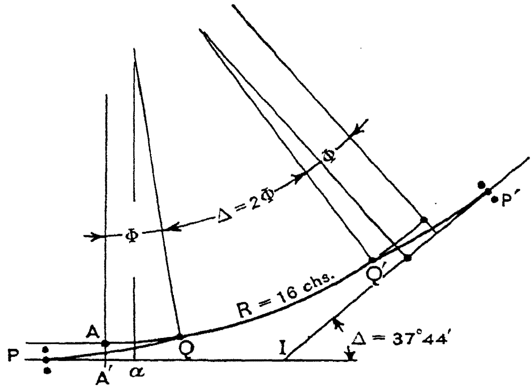


FIG. 64.

Whence the following chainages of the ruling points :

Point intersection I -	12 m.	77·846 ch.
Total tangent length		7·482
Point spiral P - -	12	70·364
Length L - -		4·000
Point spiral-curve Q -	12	74·364
Length circular arc -		6·536
Point curve-spiral Q'	13	0·900
Length L - -		4·000
Point tangent P' -	13	4·900

Peg at 12 m. 76 ch. is on circle, 1·636 ch. from Q.

$\text{Sin } \delta = \dfrac{1{\cdot}636}{2 \times 16} = 0{\cdot}051125$; and tangential angle $\delta = 2° \ 56'$.

Peg at 13 m. 3 ch. is on second spiral $Q'P'$, 1·900 chs. from P' with deflection angle $\omega = 143\frac{1}{4} \left\{\dfrac{1{\cdot}900}{4}\right\}^2 = 32{\cdot}32'$, since $\dfrac{\omega}{\Omega} = \dfrac{\lambda^2}{l^2}$.

1st spiral set out from tangent at P ; 2nd (preferably) likewise from tangent at P'. Tangent at Q established by angle 2Ω from long chord PQ, advisably sighting P with vernier initially at 2Ω, since the curves bear to the left.

The following two examples introduce cases of amending existing curves, or " spiralling old track ", the necessary shift being obtained (1) by sharpening the existing curve without changing the position of the central portion, and (2) by shifting the main curve and sharpening its radius.

Example†. In improving an existing railway curve by inserting transition curves, 4 chains in length, 6 chains of the existing 25 chains radius curve are taken up at each end and replaced in part by curves of sharper radius.

Determine the radius of the sharpened curves, also the total centre-line length of track to be relaid. (U.L.)

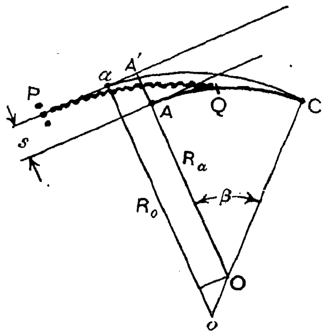


FIG. 65.

Let R_0 and R_a be the original and amended radii respectively, and let $s=$ the shift $=L^2/24R_a=\dfrac{16}{24\times 25}=0{\cdot}0267$ ch. approx. (Fig. 65).

From the figure,

$$s=(R_0-R_a)\text{ vers }\beta. \quad \text{.........(1)}$$

But $\beta=\dfrac{ac}{R_0}$ rad. $=0{\cdot}24$ rad. $=13°\,45'$,

and $\text{vers }\beta=0{\cdot}02866.$

On substituting in (1) for vers β and $s=\dfrac{2}{3R_a}$ the following quadratic is obtained :

$$R_a^2-25R_a+23{\cdot}261=0,$$

the solution of which is $R_a=24{\cdot}032$ ch.

(If the quadratic were avoided by using the approximate value of $s=0{\cdot}0267$ ch., the value of R_a would be 24·07 ch.)

Now $\dfrac{AC}{ac}=\dfrac{R_a}{R_0}$; and $Ac=\dfrac{24{\cdot}032\times 6}{25}=5{\cdot}768$ ch.

The transition replaces an equal amount of circular curve,

$$AQ=2{\cdot}000\,;$$

and since the shift AA' bisects the transition PQ; the track to be relaid at each end is 7·768 ch.

Total centre-line length of track $=15{\cdot}536$ ch.

Example†. A 3° circular curve is to be replaced by one of smaller radius so as to admit transition curves 350 ft. in length, the new curve intersecting the original one with the maximum deviation from the vertex of the latter being fixed provisionally at 18 inches, measured between the centre lines.

Determine the degree of the new circular curve, and, taking this to the nearest 5 minutes, calculate the total centre-line length of track to be

relaid, given that the deflection angle (or external angle of intersection of the tangents) is 44° 30′.

N.B. The shift for the amended radius may be taken equal to that of the original radius. (U.L.)

Let oa be the original radius R_0, OA, the amended radius R_a, Cc the central movement h, and aa' the shift s (Fig. 66).

$$R_0 = s + R_a + x \cos \tfrac{1}{2}\Delta, \quad ......(1)$$

$$h = R_a + x - R_0, \quad ....(2)$$

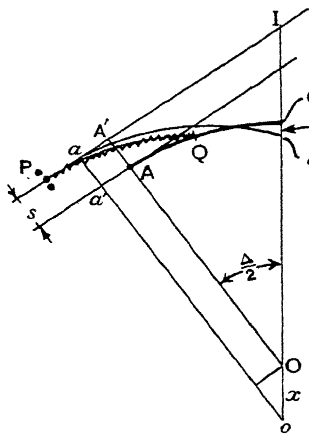


FIG. 66.

On substituting for x in (1),

$$h \cos \tfrac{1}{2}\Delta + s = (R_0 - R_a) \text{ vers } \tfrac{1}{2}\Delta. ...(3)$$

Also $s = L^2/24R_a$, but (avoiding the quadratic) may be taken here as $L^2/24R_0$.

$$R_0 = \frac{50}{\sin 1\tfrac{1}{2}°} = 1909{\cdot}85 \text{ ft.}$$

$$= \frac{5730}{D} \text{ nearly} = 1910 \text{ ft.}$$

$$s = \frac{350}{24} \times \frac{350}{1909{\cdot}9} = 2{\cdot}67 \text{ ft.}$$

Hence (3) becomes

$$1{\cdot}50 \cos 22°\ 15' + 2{\cdot}67 = (1909{\cdot}9 - R_a)(1 - \cos 22°\ 15'),$$

leading to $R_a = 1855{\cdot}35$ ft.

To the nearest 5 mins.,

$$D = 3°\ 5', \text{ and } R_a = \frac{50}{\sin 1°\ 32\tfrac{1}{2}'} = 1858{\cdot}73 \text{ ft.}$$

(Incidentally, h reduces to

$$\frac{(51{\cdot}12)(0{\cdot}07446) - 2{\cdot}67}{0{\cdot}92554} = 1{\cdot}23 \text{ ft.} = 14\tfrac{3}{4}''\text{)}.$$

The total length l to be relaid will be $2(PR + RQ)$, since the half transition RQ is precisely equal to the portion of circle AQ which it replaces.

$$l = 2\left(\tfrac{1}{2}L + \frac{R_a}{57{\cdot}3}\,\tfrac{1}{2}\Delta\right), \text{ and since } \tfrac{1}{2}\Delta = 22{\cdot}25°,$$

$$= 2\left(175 + \frac{1858{\cdot}73 \times 22{\cdot}25}{57{\cdot}3}\right) = 1793{\cdot}52 \text{ ft.}$$

Example††. The following are the total co-ordinates of points P, Q, R, S, on the centre line of a railway, Q and R being the intersection points of the tangents of circular curves of equal radii, R chains.

At each end of the circular curves, transition curves are inserted, the intermediate pair having a point of reversed curvature on QR, so that the straights PQ and RS are connected by a composite reverse curve.

Point	Total co-ordinates	
	Latitude (chs.)	Departure (chs.)
P	126·520 N.	452·640 E.
Q	129·291 N.	473·456 E.
R	163·926 N.	496·745 E.
S	163·612 N.	520·743 E.

(U.L.)

Calculate the common radius R of the main curves, given that the length of transition is $\sqrt{R}$ chains.

Bearings : $\tan \beta_1 = \dfrac{20{\cdot}816}{2{\cdot}771} = \text{N. } 82° \; 25' \text{ E.}$

$\tan \beta_2 = \dfrac{23{\cdot}289}{34{\cdot}635} = \text{N. } 33° \; 55' \text{ E.} \quad \tan \beta_3 = -\dfrac{23{\cdot}998}{0{\cdot}314} = \text{S. } 89° \; 15' \text{ E.}$

Intersection angles $\Delta_1 = 48° \; 30'$,

$\Delta_2 = 56° \; 05' + 45' = 56° \; 50'$ (Fig. 67).

Length QR : $QR \cos 33° \; 55' = 34{\cdot}635$; and $QR = 41{\cdot}737$ chs.

Shift $s = \dfrac{L^2}{24R} = \dfrac{1}{24}$ ch. $= 0{\cdot}042$ ch.

$$\begin{aligned} QR = 41{\cdot}737 &= (R + 0{\cdot}042) \tan \tfrac{1}{2}(48° \; 30') + 2(\tfrac{1}{2}\sqrt{R}) \\ &\quad + (R + 0{\cdot}042) \tan \tfrac{1}{2}(56° \; 50') \\ &= (R + 0{\cdot}042)(0{\cdot}45047 + 0{\cdot}54107) + \sqrt{R} \\ &= 0{\cdot}991541 R + 0{\cdot}041645 \sqrt{R}, \end{aligned}$$

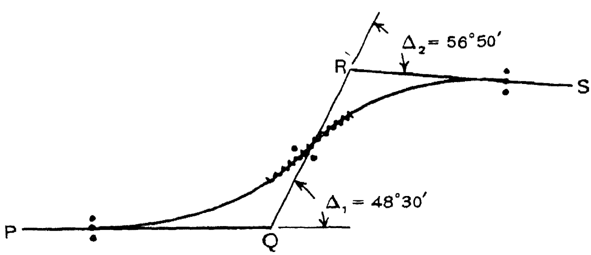


FIG. 67.

or $R + 1{\cdot}00853\sqrt{R} = 42{\cdot}05106$; $(\sqrt{R} + 0{\cdot}50426)^2 = 42{\cdot}05106 + 0{\cdot}25428$

$$= 42{\cdot}30534.$$

$$\sqrt{R} = -0{\cdot}05043 \pm 6{\cdot}5043 = 6 \text{ chains.} \quad R = 36 \text{ chains.}$$

Example†. A road bend which deflects 55° is to be designed for a maximum speed of 65 m.p.h., a maximum centrifugal ratio of $\frac{1}{4}$, and a maximum rate of change of acceleration of 1 ft. per sec.^{3}, the curve consisting of a circular arc combined with two clothoid spirals. Calculate (*a*) the radius of the circular arc, (*b*) the requisite length of transition, and (*c*) the total length of the composite curve. (U.L.)

$$65 \text{ m.p.h.} = 95{\cdot}33 \text{ f.p.s.} ; \ \frac{v^2}{gR} = \tfrac{1}{4} ; \ (95{\cdot}33)^2 = 8$$

$$= 1128{\cdot}71 \text{ ft., say, } 1130 \text{ ft.}$$

$$L = \frac{v}{R} = 8{\cdot}05v \text{ ft./sec. units} \quad \frac{95{\cdot}33 \times 95{\cdot}33 \times 95{\cdot}33}{1130} \quad 761{\cdot}3 \text{ ft., say } 760 \text{ ft.}$$

$$= \frac{L}{2R} \qquad = 1719 \frac{L}{R} \text{ mins.} = 19^\circ\ 16{\cdot}14'.$$

Central angle, $\Delta - 2\Phi = 55^\circ\ 00' - 38^\circ\ 32{\cdot}28' = 16^\circ\ 27{\cdot}72' = 987{\cdot}72'$.

$$\text{Length of circular arc} = \frac{R(\Delta - 2\Phi)}{2 \times 1719} = \frac{1130 \times 987{\cdot}72}{3438} = 324{\cdot}64 \text{ ft.}$$

Total length of composite curve $= 2\,(760) + 324{\cdot}64$ ft. $= 1844{\cdot}64$ ft.

Example††. A road transition between two straights consists of a pair of clothoid spirals meeting at a common tangent point, the second straight deflecting to the right at an angle of 19° 6′ and a running distance of 1192 ft.

The curves are to be designed for a maximum speed of 70 m.p.h., a limiting centrifugal ratio of $\frac{1}{4}$, and a maximum rate of change of acceleration of 1 ft. per sec.^{2} in a sec. (*a*) Calculate the chainages of the tangent points and the point of compound curvature, and (*b*) give an equation for the angles to be set out from the tangent in laying down the curve. (U.L.)

Here either (1) the centrifugal ratio, or (2) the comfort condition, must control.

$$(1) \ \frac{V^2}{gR} = \tfrac{1}{4}, \text{ where } V = 102{\cdot}7 \text{ ft./sec.} ;$$

$$\text{and minimum radius } R = \frac{4V^2}{32{\cdot}2} = 1304 \text{ ft.}$$

$$\text{Now } \Phi = \frac{19{\cdot}1}{2 \times 57{\cdot}3} = \tfrac{1}{6} \text{ rad., and } L = 2R\Phi = 2 \times 1304 \times \tfrac{1}{6} = 434{\cdot}7 \text{ ft.}$$

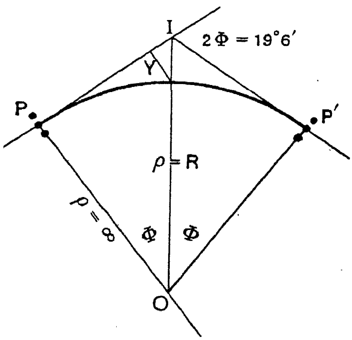


Fig. 68.

(2) $\dfrac{V^3}{LR}=1$, and $L=\dfrac{V^3}{R}=2R\Phi$;

whence $R=\sqrt{\dfrac{V^3}{2\Phi}}=10^3\sqrt{\dfrac{(1\cdot027)^3}{2\times\frac{1}{6}}}$

with $L=2\times1800\times\frac{1}{6}=600$ ft.

Obviously the point O in Fig. 68 is not a centre of curvature. Also the tangent lengths T should not be calculated from $T=Y\cot\frac{1}{3}\Phi+Y\tan\Phi$ with the basic value of $Y=\dfrac{L^2}{6R}$.

Hence
$$T=X+Y\tan\Phi$$
$$=L\left(1-\tfrac{3}{5}\frac{s}{R}\right)+\frac{L^2}{6R}\left(1-\tfrac{3}{7}\frac{s}{R}\right)\tan 9^\circ\,33'$$
$$=598\cdot34+5\cdot60=603\cdot94\text{ ft.}$$

Chainage of P $=1192-603\cdot94=588\cdot06$ ft.

„ „ $P'=588\cdot06+1200=1788\cdot06$ ft.

QUESTIONS ON ARTICLE 5

1†. Derive an equation for setting out a transition curve by offsets from the tangent wholly, or in part, from the redundant portion of the circular curve. Also give an expression for the "shift".

2†. Derive the equation to the clothoid $\lambda=m\sqrt{\phi}$, and develop formulae relative to setting out this curve with the theodolite and chain. Also show that the index ($\frac{1}{2}$) of ϕ cannot have any other value when the spiral is used as a transition curve.

3†. In improving an existing railway curve by inserting spirals 200 ft. in length, 350 ft. of the present 4° circular curve are to be taken up at each end and replaced in part by curves of sharper radius. Determine the total centre line length of track to be relaid, assuming that the sharpened curves are taken to the nearest tenth degree. [882·9 ft.] (U.L.)

4†. A circular curve of 30 Gunter chains radius is to be replaced by one of smaller radius so as to admit transition curves $5\frac{1}{2}$ chains in length, the maximum deviation from the centre line of the original curve at its vertex being fixed tentatively at 18 inches.

Determine the radius of the new circular curve and, taking this to the nearest quarter chain, calculate the total centre-line length of track to be relaid,

given that the external angle of intersection (or deflection) of the tangents is 46°.

N.B.—The shift for the amended radius may be taken equal to that of the original radius. (U.L.)

[$R_a = 29\frac{1}{4}$ ch. ; $l = 29{\cdot}322$ ch.]

5†. The bearings of two intersecting straights on the centre line of a proposed railway are S. 64° 34′ E. and S. 18° 39′ E., the intersection occurring at 76 + 24 in 100-ft. units. A 5° circular curve and transition curves are to be inserted, the latter being $600/\sqrt{D}$ ft. in length, on the assumption that the circular curve is to carry its maximum allowable speed.

Tabulate the data relative to Stations 71 and 72, and the junctions of the transition curves with the circular arc. (U.L.)

[$\omega = 17' \, 10''$; 1° 11′ 16″. 95·56 ft.; 195·56 ft. P.S. = 70 + 4·44 ; P.S.C. = 72 + 72·77 ; P.C.S. = 79 + 22·77 ; P.T. = 81 + 91·10.]

6†. Show that a rational rule for the length L of a transition curve is of the form $L = av^3/R$, where a is a constant and R and v respectively the radius and speed on the curve.

Deduce accordingly rules for the length of transition in the British and American systems, making the following assumptions :

(*a*) Maximum allowable speeds on main curves of $11\sqrt{R}$ m.p.h. and $100 \div \sqrt{D}$ m.p.h. respectively, R being the radius in chains and D the degree per 100 ft. unit.

(*b*) Limiting rate of increase of radial acceleration, 1 ft. per sec.² in a sec.

Reducing these rules to the nearest working approximations, calculate the tangential (or deflection) angles (1) for spirals of $\frac{1}{2}$ chain chords to a curve of 40 chains radius, and (2) for ten equal chords of a spiral to a 4° curve, the main curves carrying the maximum allowable speed in each case. (U.L.)

[$L = \sqrt{R}$ chs. (550 to 600)/$\sqrt{D}$ ft. ; (1) 2·26′ ; (2) 0·55′ to 0·60′]

7†. Show that a cubic parabola is suitable for a railway transition curve, and explain clearly how the clothoid becomes a cubic parabola when set out normally with the theodolite and chain, the intrinsic equation of the curve being $\lambda = m\sqrt{\phi}$, where m is constant and λ and ϕ are the co-ordinates of a point with respect to an origin assumed at the point of tangency of the spiral with the main tangent. (U.L.)

8†. State concisely why transition curves are inserted as approaches to circular railway curves ; and show that a theoretically perfect transition curve is to be found in the spiral

$$\lambda = m\sqrt{\phi},$$

where m is a constant, and λ and ϕ the intrinsic co-ordinates of points on the spiral.

What lengths of transition would you use in connection with a circular curve of 36 chains radius when the maximum permissible speed is (*a*) 40 m.p.h., and (*b*) 66 m.p.h., the latter being the limiting speed for the radius. (U.L.)

[(*a*) Uniform approach with $k = 500$, 1·68 chs., or by Shortt's rule for time approach, 1·29 chs. for (*a*) and 6 chs. for (*b*).]

9. Two tangents intersect at $200+00$, the deflection angle being $30°$. Using four-figure mathematical tables :

(*a*) Calculate the radius, length, and subtangent of a $3°$ circular curve.

(*b*) Make all necessary calculations to set out this curve by deflection angles and also by offsets from the tangent.

(*c*) Make all necessary adjustments to allow for the placing of a transition curve of 400 ft. at each end of a $3°$ circular curve, and all calculations to enable the transition curve to be set out by offsets from the original tangent.

What angle does the common tangent to the transition curve and the circular portion make with the original tangent direction, and how is this common tangent placed in the field to enable the circular portion to be set out from it? (U.B.)

[(*a*) $R=1914{\cdot}6$ ft. $L=1000$ ft. ; $T=513{\cdot}0$ ft.

(*b*)	195	196	197, etc.
δ	12′	1° 42′	3° 12′ at 1° 30′ to 15°
Offset :			
y	0·04	3·37	11·91 ft., etc.
x	12·8	113·4	213 ft. to 513 ft.

(*c*) Total tangent length : $513+0{\cdot}93+200=713{\cdot}93$ ft.
Offset : $y'=0{\cdot}2176\times 10^{-6}x^3$ to $y'=13{\cdot}92$ ft. when $x=400$ ft.]

10. At a road bend the total deflection angle made by the straights is $75°$. The bend is to be formed by an $8°$ curve having a total deflection of $48°$ and two spiral transition curves.

Find (*a*) the lengths of the spiral curves, (*b*) the distance of the centre point of the circular curve to the intersection point of the straight.

For the basic spiral, assume $x=s-\dfrac{s^5}{40}$ and $y=\dfrac{s^3}{6}$.

(Slide rule calculations will be accepted.) (U.G.)

[(*a*) 338 ft. ; (*b*) 195 ft.]

11. What is a transition (or easement) curve? Why is it used? Define "shift" of a curve. Draw two tangents and show a circular curve and two transition curves connecting the tangents, marking the "shift" on your sketch. How may the transition curve be set out? (U.D.)

12. What are the reasons for inserting a transition (or easement) curve between a straight and circular curve on a railway? What is the type of transition curve most commonly used and how is its length determined? What is meant by "shift"? Explain how the transition curve may be set out by offsets or deflection angles. (I.C.E.)

13. Explain the reasons for the desirability of introducing a transition curve between a tangent and a circular railway curve.

In a certain case the radius of the circular curve is half a mile, and the maximum speed on it is 50 miles per hour. Describe how you would decide upon a suitable length for the transition curve, and state what length you would recommend. (I.C.E.)

[2 to $2\frac{1}{2}$ chs.]

14. Part of a road straight is to be displaced 200 ft. and connected at the end to the original line by reversed curves formed of double spirals. The distance available for each reverse curve, measured in the direction of the straights, is 1000 ft. Calculate the tangent lengths, curve length, and minimum radius of curvature, and the speed corresponding to a rate of radial acceleration of 1 ft./1 sec.^{3}.

Explain how the curves may be set out. (U.G.)

$$x = s - \frac{s^5}{40}; \quad y = \frac{s^3}{6}.$$

[$T = 238{\cdot}7$ ft.; length $= 1009$ ft.; $R = 1697$ ft.; $V = 81{\cdot}6$ m.p.h.]

15. Prepare a table for setting out any type of non-circular highway curve with which you are acquainted. Only sufficient entries to show the method clearly are required. The intersection angle is 90 degrees, and the minimum radius of the curve is 500 feet. (I.C.E.)

16. What are the reasons for introducing transition curves between straights and circular curves?

Describe the transition curve that is commonly used on the South African Railways, and calculate data for the setting out of a suitable transition curve of this type for a 4° curve.

Explain also why this is not a satisfactory curve for sharp bends and describe in outline a suitable curve as used in modern road design for high-speed traffic.

With a ruling grade of 1 in 100 state how the grade would be compensated on the above-mentioned curve. (U.C.T.)

17. A road curve of 600 feet radius is to be set up to connect two tangents. The maximum speed on this part of the road will be 44 feet per sec. Transition curves are to be introduced at each end of the curve. Find a suitable length for the transition curves and calculate :

(i) the necessary shift of the circular arc,

(ii) the chainage at the beginning and at the end of the combined curve, and

(iii) the value of the first two deflection angles of the transition curve.

Angle of intersection is 63° – 36′.

The rate of change of acceleration is 1 ft. per sec. per sec. in a sec. The chainage of the intersection point is 3640·6 ft.

If the width of the road is 30 feet, what height of banking should it be given? (T.C.C.E.)

[$L = 142$ ft., $s = 1{\cdot}40$ ft.; $T.P._1$, 3196·72; $T.P._2$, 4004·93; 67·25′ per (100 ft.)^2; 3 ft.]

18. Derive an expression for the length and the shift of a transition curve required for a first-class railway track. (T.C.C.E.)

19. Explain briefly with a diagram how a transition curve is set out by means of a theodolite.

The tangents to a railway curve meet at an angle of 148°. The curve to be chosen has to pass near a point P which is 60 feet from the point of inter-

section on the line bisecting the angle of 148° between the tangents. Find to the nearest half degree the degree of a suitable curve.

If the speed of the train is limited to 60 miles an hour and the rate of change of acceleration to 1 ft. per sec. per sec. in a sec., calculate the necessary data for setting out the transition curve. (T.C.C.E.)

ARTICLE 6: VERTICAL CURVES

A vertical curve is used at the intersection of two different or contrary gradients in order to avoid sudden changes in slope on railways and highways. On railways it is customary to introduce a parabolic curve whenever the change in rate of gradient exceeds about 0·2 per cent, but a much higher figure is permissible in the case of highways, where the range of vision is an important factor in determining the length of the curve L.

In applying the parabola $y=cx^2$, the first gradient *produced* is the abscissa and, since the effect of this gradient is relatively small, the ordinates are measured vertically, usually as staff readings taken with a level. Also since it is convenient to set out the curve with a number $(2n)$ of equal chords of length l, n on each side of the centre, the practical form $y=kN^2$ is more convenient, N being counted from 0 at the beginning of the curve A. In this connection, g and g_1 are the gradients and r and r_1 the corresponding rises and falls per chord length l, r and r_1 being figured plus or minus according as they represent rises and falls and thus avoiding confusion in the six possible cases that may arise.

Now it follows from Fig. 69 that the elevation of G is $2nr$, and the terminal ordinate $y_B=$ elev. $G-$ elev. $B=n(r-r_1)$, with $N=2n$ in $y=l$

Thus $k=\dfrac{(r-r_1)}{4n}$.

Since $AE=EB$, CE is a diameter of the parabola, and $CD=DE$ accordingly, while if the elevation of A be assumed zero,

$$\text{elev. } C=nr\ ;\ \text{elev. } B=\text{elev. } C+nr_1=n(r+r_1)\ ;$$

$$\text{elev. } E=\tfrac{1}{2}(\text{elev. } A+\text{elev. } B)=\tfrac{1}{2}n(r+r_1)\ ;\ \text{elev. } D=\text{elev.}$$

The elevation of D may also be found by subtracting algebraically n^2k from the elevation of C; that is,

$$\text{elev. } D=nr-n^2k. \qquad .\text{(ii)}$$

Whence $$k=\frac{r-r_1}{4n}.$$

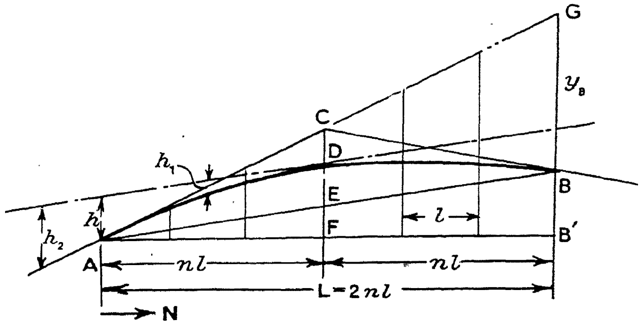


FIG. 69.

In practice, tangent corrections are commonly employed, though some prefer to work with chord gradients, establishing successive chords of the curve from $g = r - (2N - 1)k$.

In the case of highways the range of visibility can be investigated by assuming a parallel to AB through the vertex D, the height of the eye being h at A and B, h_1 between A and D, and h_2 preceding A, being measured down to the gradient AC in the last case. Steep gradients involve a more precise analysis.

(i) **Sight length equal to length L of the curve; (m = n).**

From
$$h = DE = n^2k = \frac{n(r - r_1)}{4},$$

it follows that
$$n = \frac{4h}{(r - r_1)}.$$

(ii) **Sight length less than L; (m < n).** Assuming a ratio m of half the length of the curve, $N = (n - m)$ with N counted from A.

$$h_1 = m^2k = \frac{m^2}{4n}(r - r_1),$$ m being measured from the centre outwards.

(iii) **Sight length greater than L; (m > n).**

$$h_2 = DE + (m - n)(r - r_1) = \frac{(2m - n)}{4}(r - r_1),$$

from which n is determinate for given values of h, r and r_1.

Thus for $l = 50'$ with h, h_1, and $h_2 = 4{\cdot}5'$, $g = -g = 1 : 50$, $r = -r_1 = 1$, and respective sight lengths of L, $\frac{1}{2}L$, and $2L$,

(i) $n = 9$ and $L = 900'$; (ii) $n = 36$ and $L = 3600'$;

and (iii) $n = 3$ and $L = 300'$.

Some authorities suggest that a highway curve should be used when the change of gradient exceeds 5 per cent, and that the absolute minimum lengths should be 350 ft. in hilly country up to 800 ft. in heavy traffic.

Example†. The following notes show the surface and formation levels in the vicinity of Sta. 125, the proposed vertex of a vertical parabolic curve which is to be laid down in six 50-ft. stations :

Station :	123·5	124	124·5	125	125·5	126	126·5	(100 ft.
Surface level :	78·5	77·9	77·5	77·3	76·7	74·9	73·6	units)
Formation level :	75·7	75·7	75·7	75·7	74·5	73·3	72·1	

Show in tabular form for an assumed height of collimation of 82·92 the staff readings that must be obtained if pegs are to be driven with their tops consistently 3 ft. above the formation of the curve.

Calculate also the excess earthwork introduced by the curve, given that the formation width is 30 ft. and the side slopes 1 to 1, the ground being level across. (U.L.)

Here the rates of gradient r and r_1 per 50-ft. station are 0 and $-1{\cdot}2$, giving $k=0{\cdot}10$ in $y=kN^2$, y being the tangent correction to the first grade ($r=0$) produced.

Station	Formation level	Tangent correction	Curve formation	Staff reading
123·5	75·70	0·0	75·70	4·22
124	75·70	0·10	75·60	4·32
124·5	75·70	0·40	75·30	4·62
125	75·70	0·90	74·80	5·12
125·5	74·50	1·60	74·10	5·82
126	73·30	2·50	73·20	6·72
126·5	72·10	3·60	72·10	7·82

Additional earthwork volume $=3249{\cdot}5$ cu. ft. $=120{\cdot}4$ cu. yds.

Example†. Determine the length L of a vertical parabolic curve which is to be set out in 50-ft. stations on a highway between an ascending gradient of 1 : 40 and a descending gradient of 1 : 30, the specified range of vision from a height of 4′ 6″ being equal to the length L.

Also ascertain the running distances and reduced levels of the following points on the curve, given that these values for the point of intersection of the gradients are respectively 3645 and 78·50 ft.

(*a*), (*b*) Beginning and End of Curve. (*c*), (*d*) Vertex and Summit of Curve.

Here $h = DE = n^2k = \dfrac{n(r-r_1)}{4}$; $n = \dfrac{4h}{r-r_1}$ (Fig. 69).

$$r = 1{\cdot}25 \;;\; r_1 = -1{\cdot}67 \;; \qquad n = \frac{18}{1{\cdot}25 + 1{\cdot}67} = 6{\cdot}2 \text{ (50 ft. chds.)}$$

Say, $k = \dfrac{r-r_1}{4n} = \dfrac{2{\cdot}92}{24} = 0{\cdot}1217$ with $n = 6$ and $L = 600$ ft.; $y = N^2k$.

Slope at summit referred to x-axis along first gradient :

$$\frac{dy}{dN} = r = 1{\cdot}25 = 2Nk \;;\; N = \frac{1{\cdot}25}{0{\cdot}2434} = 5{\cdot}136 \text{(50 ft. chds.)} = 256{\cdot}8 \text{ ft.}$$

(Obviously this will not be given by equating the chord gradient to zero ; that is, $r - (2N-1)k = 0$, giving $N = 281{\cdot}75$ ft., which corresponds to 25 ft. ahead.)

(*a*)	3345	ft. Beginning of Curve :	$78{\cdot}50 - 7{\cdot}50 = 71{\cdot}00$	Reduced
(*b*)	3945	,, End ,,	$78{\cdot}50 - 10{\cdot}00 = 68{\cdot}50$	Level
(*c*)	3645	,, Vertex ,,	$78{\cdot}50 - 36 \times 0{\cdot}1217 = 74{\cdot}12$	
(*d*)	3601·8	,, Summit ,,	$71{\cdot}00 + \frac{1}{40}(256{\cdot}8) - (5{\cdot}136)^2 0{\cdot}1217 = 74{\cdot}21.$	

QUESTIONS ON ARTICLE 6

1†. A stake C indicates the intersection of two descending gradients PC and CQ, of 0·44 per cent and 1·08 per cent respectively, the stakes P, C, and Q being driven 3 ft. above formation. These gradients are to be eased with a vertical parabolic curve which is to pass 0·32 ft. below the gradient at the intersection at C, the reduced level of the top of the stake at P being 105·00.

Calculate by the method of chord gradients the staff readings to be obtained on the tops of stakes driven 3 ft. above formation at 50 ft. stations on the curve from a collimation height of 108·91 above datum.

[Here $r = -0{\cdot}22$; $r_1 = -0{\cdot}54$; $n = \dfrac{4y}{r-r_1} = \dfrac{4 \times 0{\cdot}32}{-0{\cdot}22-(-0{\cdot}54)} = 4.$

Chord gradients : $\varsigma = r - (2N-1)k$, where $k = 0{\cdot}02$.

Stake :	P	(A)	1	2	3	$4(C)$	5	6	7	$8(B)$	Q
Staff :	3·91	4·13	4·37	4·65	4·97	5·33	5·73	6·17	6·65	7·17	7·71.]

2†. The following are the surface and formation reduced levels in the vicinity of Station 242, the proposed vertex of a vertical curve which is to be laid down in six 50-ft. stations :

Station :	240·5	241	241·5	242	242·5	243	243·5	(100 ft.
Surface :	76·1	75·6	75·2	75·1	74·3	72·6	71·3	units)
Formation :	73·4	73·4	73·4	73·4	72·2	71·0	69·8	

Record the calculations in tabular form, and show, for an assumed height of collimation of 80·82, the staff readings which must be obtained if the pegs are driven with their tops consistently 3 ft. above formation. (U.L.)

[Staff readings : 4·42 ; 4·52 ; 4·82 ; 5·32 ; 6·02 ; 6·92 ; 8·02]

3†. Derive an expression for the tangent corrections to the first gradient produced in setting out vertical parabolic curves between two different gradients on the centre line of a proposed railway.

4. (*a*) If the maximum cant allowed is 6 inches and from C. to C. of rails is 59 inches, obtain an expression for the length of a transition curve in terms of the radius of the circular portion of a railway curve. If the radius of the circular portion is 4,000 feet, what is the length of the transition curve?

(*b*) State the recommendation of the American Railway Engineering Association for the length of a vertical curve (i) on summits, (ii) on sags.

(*c*) A vertical parabola, 400 feet long, is to be put in between a 2 per cent up grade and a 1 per cent down grade which meet at a chainage of 1,000 feet, the reduced level of the point of intersection of the two gradients being 300·00 feet. Obtain the reduced levels of the tangent points and at every 50 feet along the parabola. (U.B.)

[(*a*) 375 feet by Shortt's rule, which is 750 times the cant;

(*c*) (T.P.) 296·00, 296·91, 297·63, 298·16, *298·50*, 298·66, 298·62, 298·41, 298·00 (T.P.).]

5. Write brief notes on the following:

Lemniscate Curve, Vertical Curve, Compound Curve, and Reverse Curve.

An up grade 1 in 100 is followed by a down grade 1 in 200. The reduced level of the intersection point is 450·00 feet. Calculate the necessary data for setting out the curve when it is of parabolic form. (T.C.C.E.)

[Assuming length of 400 ft., with y the offset down from the first gradient produced:

	0,	50,	100,	150,	200,	250,	300,	350,	400 ft.
y:	0,	0·05,	0·19,	0·42,	0·75,	1·17,	1·69,	2·30,	3·00 ft.]

ARTICLE 7: TUNNELLING

It would be outside the scope of a book of this nature to review the various topographic and economic factors which decide the construction of a tunnel in preference to a cutting, etc. Other factors eliminated, a depth of about 60 ft. determines the economic depth of a cutting, particularly if the ground rises for a considerable distance afterwards. The survey work is so closely allied to specialised operations that it is necessary for the student to consult the various works on railway engineering or tunnelling. Hence the following problems necessarily lack many important features that appear in practical data. In order to give con-

tinuity to the subject, the various operations are treated collectively, comprehending (*a*) surface survey, (*b*) transferring the alignment underground; (*c*) underground setting-out; and (*d*) levels in tunnels.

The term " surface survey " is used to suggest that the alignment is not necessarily over the centre line on the surface; but since the conditions of such survey may influence both the succeeding operations, these will be considered together in four general cases, the levelling work being deferred to the end of this article.

Case 1. When the depth is not great and surface alignment is possible. Here the centre line of the proposed tunnel is accurately marked out on the surface of the ground and is transferred through shafts to the underground survey. The operations may be varied according as (i) many shafts may be sunk, or (ii) only one shaft is practicable on account of the configuration of the ground.

(i) When a large number of shafts may be sunk. Here Simms' method may be employed, which briefly is as follows:

Timber baulks, *AB* are fixed laterally, spanning the shaft near its edges, and the surface centre line is transferred to these, preferably on plates drilled with holes for suspending the wires of the plumb bobs, *a*, *b*, which weigh 20–30 lb., the movement of these being damped by immersion in muddy water or treacle. The line is continued underground, as shown in Fig. 70, the alignment being marked on centre-punched nails or dogs driven into convenient byats of timber (*c*). Lamps or bobs (*d*) may be suspended from these nails.

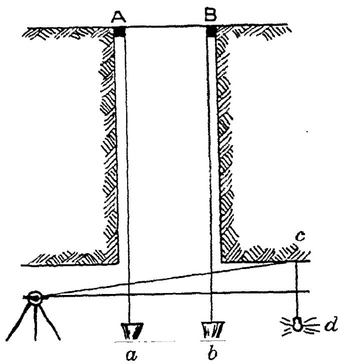


FIG. 70.

When tunnels are iron-lined, wooden wedges may be driven between the joints of the segments, the line being marked on nails in the wedges. The same method may be used in brick-lined tunnels, or cross byats may be built in the brickwork for carrying the marks.

Where the nature of the work will permit, the marks may be set on brass nails in the heads of stout stakes driven into the invert of the tunnel. The stakes should be surrounded by brickwork parged over with cement flush with the top of the peg. Underground marks of this kind allow the instrument to be set up over them, which is preferable to lining in by means of two plumb-bobs.

(ii) When only one shaft can be sunk. The case of a single shaft arises when the depth is so great that the cost of sinking is prohibitive, and also when the vibrations caused by pumping machinery, in or near the shaft,

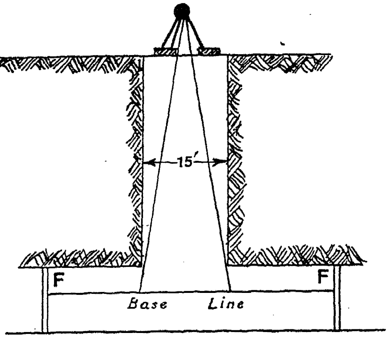


FIG. 71.

may impair the use of vertical wires. One method of procedure is shown in Fig. 71, where heavy baulks for stationing the theodolite are fixed across the mouth of the shaft, separate baulks being placed for the observer. The theodolite is levelled up on the surface centre line, and downward sights are taken as indicated; the hollow-centre mining theodolite is frequently an asset in the work. Underground, the headings are driven 40 to 50 yards in approximately the right direction, and timber frames, F, F, are fixed in the headings, about 100 yards apart accordingly.

These frames are provided with horizontal screws, which carry a wire over their threads, the wire being heavily weighted at the ends. This *base wire*, as it is called, is put in line by means of the theodolite, which is used both face left and face right; and with a base of 100 yards, the wire can be set with great accuracy, particularly if it is illuminated with electric light, as in the case of the Severn Tunnel.

Limitations are imposed upon the method by the difficulties of sighting the wire at great depths owing to falling water and dirt.

Two matters of general interest may be considered opportunely at this juncture in regard to the underground work.

Underground sights. Various means of illuminating the plumb wires or reference marks have been employed, and these are primarily dependent upon the distance. (*a*) For *short* sights, an excellent arrangement is that of sighting the plumb line against a white background of oiled paper, illuminated from behind. (*b*) For medium sights, candles with appropriate holders have been used, also sighting lamps (battery or oil) with cross-lines and holders. (*c*) For *long* sights, the Argand oil lamp, 40–50 c.p., carried in an adjustable metal frame, has proved successful, though in recent work this has been improved upon. (*d*) For *floor* stations, the plumb line illuminated as in (*a*) may be used, or a vertical illuminated slit, or the plummet lamp.

In connection with these, the ordinary means of axis illumination may be employed, or, failing these, the front reflector (see p. 266).

When alignment follows from two plumb lines, it may be necessary (*a*) to have the suspensions at different levels and the wires of different lengths, (*b*) to insert links in the nearer wire in order that the distant wire can be seen, or (*c*) to use a thick and a thin wire, the difference being emphasised by the interposition of a white card.

Aligning the theodolite. There is diversity in practice as to the position of the theodolite at both the surface and the bottom of the shaft. Some engineers station the instrument at a distance from the near wire equal to the distance between the wires, alignment being judged by eye and the change in focussing. Others prefer to set up at 30–50 ft. from the nearer wire, using the same focus, which, however, must be altered when sighting on permanent marks or stations. Usually the sights are taken with both faces of the instrument, primarily to eliminate the effect of errors of adjustment.

Oscillations of the wires can be averaged out by eye when small, but when large the extreme readings are averaged by means of a scale behind each wire, or by using a diaphragm provided with a fine scale.

An alternative method of connecting the surface and underground surveys is to supersede direct alignment by the following process, which is used extensively on the Continent of Europe.

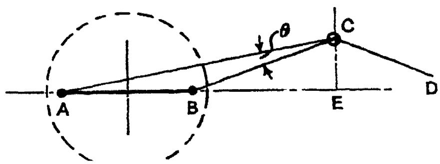


FIG. 72.

Weisbach triangle. Here the theodolite is set up at C, near, and almost in line with AB, so that the effect of error in BC or AC on θ is least when θ is least, $\theta = ACB$ being usually less than 30′ (Fig. 72).

The angle θ is measured very accurately, as also are the distances AC, BC, and, as a check, AB. In order to get a check on θ, the angles ACD, BCD, which AC and BC make with any line CD, may be measured, θ being found by subtraction.

If desired, the offset CE from the vertical plane of AB can be calculated, and a line parallel to the centre line set out.

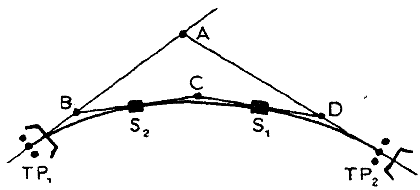
The weakest point in the method is the fact that lines and angles are measured from unsteady points A and B, rendering it necessary to take the extreme positions of each plumb wire.

Case 2. When shafts can be sunk, but surface alignment is impossible. The examples of this category arise mainly from the difficulties encountered in towns, in that either (*a*) it is impossible to set out the line on the surface, or (*b*) the shafts cannot be placed on the centre line of the tunnel.

(*a*) The present case differs from Case 4 (p. 150) in that shafts are practical and the length involved relatively small. The latter fact allows

the use of a precise traverse, triangulation being precluded in built-up areas. Extreme care and due regard of corrections must be exercised in the steel tape measurements, and the angles must be repeated with both faces of the theodolite. Effort should be made to embody a closed traverse in order that the corrections may be applied or, failing this, the traverse should be based upon two or more accurate, pre-determined points, preferably triangulation points. From the corrected co-ordinates of the traverse it will then be possible to calculate the position, direction, and chainage of the centre line at any specified point.

(*b*) In tunnelling under towns it may also be inexpedient or impossible to locate the shaft on the centre line, as, for example, when the tunnel is under a major highway. Whenever it is thus necessary to use an eccentric shaft, the latter is connected with the tunnel by means of an adit, or entry tunnel. Normally the plumb lines are suspended in the shaft on some known azimuth, so that their line can be produced through the adit into the tunnel, where the theodolite is set up in line at a distance from the wires equal to their distance from the centre line of the tunnel, which is fixed by setting off the appropriate angle.



G. 73.

Tunnelling on curves. Curves should be avoided in tunnelling whereever possible, though the inherent difficulties are frequently encountered near the extremities.

It is desirable, though not essential, to set out the curve on the surface, but if this is impracticable, sufficient measurements should be made on the surface to fix the positions of the tangent points exactly. Every case will introduce its peculiar difficulties, for the field is wide, ranging from the comparative simplicity of large radii to the cramped conditions of the sharp radii of the Underground lines of London.

Shafts may be sunk on the centre line to facilitate progress, allowing excavation to be carried on from four or more points simultaneously. Subsidiary surface tangent lines may be laid down at the centres of the shafts (S_1, S_2), stages (B, C, D) being erected to adjust these points, as in the case of the Fairlie Tunnel (Fig. 73).

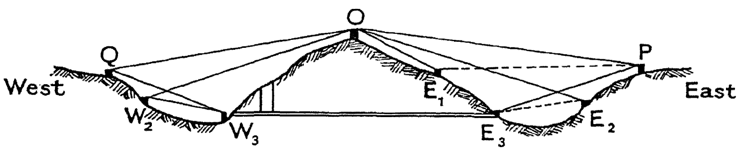
Setting out from the open ends only is a matter of little difficulty, since long bases from which to work are usually available ; but when the

setting-out has also to be conducted from shafts, the utmost care must be exercised in transferring the lines tangent to the curve from the surface to the level of excavation. Although the curve ranging follows the usual procedure, it is limited by the fact that the tangent lengths cannot be measured as on the surface, and the theodolite must be moved forward as soon as the whole chord for the portion accessible meets the side of the tunnel.

Case 3. When the depth below the surface is too great for shafts to be sunk, but surface alignment is possible.

An outline of the operation, based upon the procedure in the setting-out of the Totley Tunnel, is given with reference to Fig. 74.

Here the points O and P (or E_1) are lined over the proposed centre line with a 6″ theodolite; the point Q is then lined in with O and P (or E_1) with the instrument at O, also the points E_3 (or E_2) and W_3 (or W_2) on a level with the openings of the tunnel. Brick or concrete structures, styled **observatories**, are erected at O, P, Q, and (say) E_3 and W_3, with a separate pier for carrying the theodolite.



. 74.

Permanent centre marks establishing the line are fixed at these stations and the alignment is checked from both P and Q, E_2 and W_2 being interpolated in order to check the accuracy from the east and west headings as the work progresses simultaneously from each end. The number of permanent stations and consequent checks is determined by the length of the tunnel and the configuration of the ground.

The differences of level of E_3, W_3, etc., are ascertained by spirit levelling, which is advisably verified by trigonometrical reduction from observed vertical angles.

Although shafts are constructed at the ends for alignment purposes, these are not in the primary sense of those in the foregoing methods.

Frequently it may be necessary to retain the suspension wires in spite of the fact that their presence would interfere with the progress of operations. In order to obviate this in the construction of the Totley Tunnel, the wire was wound on a horizontal drum, with ratchet and pawl safety-device, while the point of suspension was adjustable through the medium of the nut of a horizontal screw, the latter device being carried by a frame fixed inside the edge of the shaft.

Case 4. When the depth below the surface is too great for shafts and surface alignment is impossible.

Characteristic of this case is the setting-out of alpine tunnels. After careful reconnaissance, the stations of a scheme of triangulation are established with the aim of obtaining data which will determine the direction or prolongation of the centre line. The surface survey thus reduces to a minor triangulation, and the sides are computed after the angles have been adjusted in accordance with the various equations of condition. Once the direction of the centre line is determined, the work is proceeded with on similar lines to the foregoing method.

Levels in tunnels. Wherever possible, the longitudinal section along the whole course of surface alignment is obtained, and benches are established near each shaft and at the ends of the tunnel. Careful checking of these levels is essential, since error in level may be quite as serious in effect as error in alignment.

In transferring levels underground, little difficulty is encountered at the ends of the tunnel, but at the shafts the ordinary methods are necessarily superseded by the use of steel bands, chains, or rods.

(1) When the cage travels in wooden guides, the level of a mark near the top of a guide is obtained from the nearest surface bench mark. An assistant holds one end of the tape on the mark, and the engineer descends the shaft, and marks the lower end of the tape on the guide. The assistant is then lowered, and holds his end on the mark, the process being repeated until the bottom is reached. Temporary platforms should be placed at depths of 66 ft. or 100 ft.; otherwise the engineer must be carried on a seat fixed to the winding rope.

(2) When the chain is used, it must be tested under a pull equal to its own weight. The chain is suspended from a nail of known level, and a second nail is driven within the lower handle, and just touching it. Round nails of fixed diameter d must be used, since in an apparent depth D measured in N lengths, the true depth will be $D - N(d + 2t)$, where t is the thickness of the handles.

(3) **Borcher's measuring rods** furnish the most accurate means of measuring vertical distances. These are round steel rods, up to $\frac{1}{4}''$ diameter, and 1–4 yards long, with double-nut connections for making a rod of any desired length. The upper rod is hooked for suspension, and the brass nuts are cut away in order to expose the plane ends of the rods at the joints.

(4) When the shaft is very deep, direct vertical measurement is changed to horizontal, as shown in Fig. 75, where planks are laid for the measurement. A fine steel wire, loaded with 10–30 lb., passes over a pulley Q from a windlass, the wire being in contact with horizontal threads, CC, DD at the top and bottom of the shaft. When the wire is lowered the points of contact are marked, and the wire still loaded is

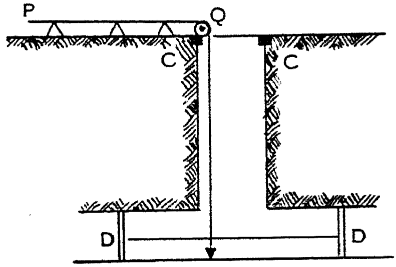


FIG. 75.

wound up and stretched on the planks, where the distance between the marks is measured. The method is rapid, simple and accurate, and elongation is not involved, since the wire is under the fixed tension in both positions.

Underground bench marks. When the reduced level of a point underground has been found, the level is set up and a bench mark is established. In early stages this may be merely a stake driven firmly near the bottom of the shaft, the position being chosen so that it will not be disturbed or hamper operations. When the lining has progressed some distance, benches are established on spikes in the brickwork, or on the flanges of the segments, as the case may be. More benches are inserted as the work progresses, one or more being kept near the working face. Since the headings are well in front of the lining, any error that may exist is allowed for by putting in a **junction gradient.** A check level is run through the tunnel from end to end after the headings from the two ends meet.

Interlacing triangles. Problems may arise in connection with tunnelling (Case 4 particularly) where it will be impossible to observe the angle adjacent to (*a*) one side or (*b*) a diagonal, the distances involved being impeded. Since precise work should introduce the log sine condition of minor triangulation, this should be applied as follows, regarding quadrilaterals as interlacing triangles without a central station.

Consider the outer angles as being divided into two parts; and, traversing the figure in the counterclockwise direction, figure the left-hand parts *even* and the right-hand parts *odd*, as in Fig. 76.

$$AB = \frac{BC \cdot \sin 4}{\sin 1} \quad \frac{CD \cdot \sin 6 \cdot \sin 4}{\sin 3 \cdot \sin 1}$$

$$= \frac{DA \cdot \sin 8 \cdot \sin 6 \cdot \sin 4}{\sin 5 \cdot \sin 3 \cdot \sin 1}$$

$$= \frac{AB \cdot \sin 2 \cdot \sin 8 \cdot \sin 6 \cdot \sin 4}{\sin 7 \cdot \sin 5 \cdot \sin 3 \cdot \sin 1}.$$

Whence: $\sin 1 \times \sin 3 \times \sin 5 \times \sin 7$
$$= \sin 2 \times \sin 4 \times \sin 6 \times \sin 8;$$

or sum of log sines L.H. angles
= sum of log sines R.H. angles.

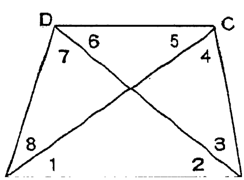


FIG. 76.

The method is used in determining the unknown angles in the following problems: In (5) α and β for 1 and 2; in (6) A_l and A_r for 8 and 1.

Example†. The following notes refer to the alignment down a shaft by means of the Weisbach triangle, A and B being the plumb wires, C and D the respective surface and underground theodolite stations, and P and Q the reference points accordingly.

Determine the angle between the reference lines CP and DQ, given that zero readings were taken on the reference points in each case.

Station	Line	Length/ft.	Angle
C	CA	13·04	72° 16′ 25″
	CB	26·50	72° 16′ 21″
—	AB	13·45	—
D	DA	14·74	176° 4′ 36″
	DB	28·20	176° 4′ 27″

(U.L.)

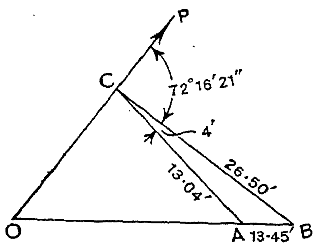


FIG. 77 (*a*).

Surface. For small angles,

$$CBA = 4'' \times \frac{13{\cdot}04}{13{\cdot}45} = 3{\cdot}9$$

Therefore $CAO = 4'' + 3{\cdot}9 = 7{\cdot}9''$,

and

$$\begin{aligned} COA &= PCA - CAO \\ &= 72^\circ\ 16'\ 25'' \\ &\quad\ \ - 7{\cdot}9'' \\ &\quad\ 72^\circ\ 16'\ 17{\cdot}1'' \end{aligned}$$

Underground. Likewise $DBA = 9'' \times \dfrac{14{\cdot}74}{13{\cdot}45} = 9{\cdot}8''$.

Therefore $DAM = 9 + 9{\cdot}8 = 18{\cdot}8''$,

and

$$\begin{aligned} DMA &= 176^\circ\ \ 4'\ 36'' \\ &\qquad\qquad\ \ 18{\cdot}8'' \\ &\quad\ 176^\circ\ \ 4'\ 17{\cdot}2'' \end{aligned}$$

But

$$\begin{aligned} COA &= \ \ 72^\circ\ 16'\ 17{\cdot}1'' \\ &\quad\ 103^\circ\ 48'\ 00{\cdot}1'' = \text{angle between } CP \text{ and } DQ. \end{aligned}$$

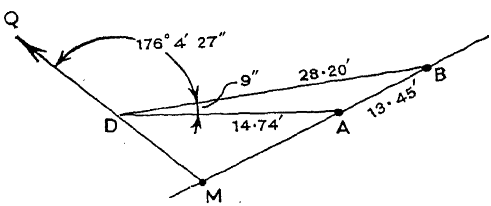


FIG. 77 (*b*).

QUESTIONS ON ARTICLE 7

1††. Write a concise account of the methods employed in setting-out the centre lines of long tunnels :

(i) When shafts can be sunk on the centre line : (*a*) single shaft ; (*b*) several shafts.

(ii) When depth is too great for shafts to be sunk, and surface alignment is (*a*) possible, (*b*) impossible.

Sketches should be given wherever desirable, and the discussion should include notes on (i) connecting the underground lines with the surface survey, and (ii) transferring levels underground, giving an example in both the cases of shallow and deep shafts. (U.L.)

2†. Describe some method that has been employed in setting-out the centre line of a tunnel on a curve.

3†. Discuss the use of the Weisbach triangle.

4†. Describe, with reference to sketches, the method used to connect an underground survey with a surface survey when only one shaft is available in the following cases :

(*a*) shallow shaft ; (*b*) when upward sights are impossible on account of wet, etc. ; (*c*) when depth is great and downward sights are unreliable, giving also the case where it is advisable to use the Weisbach triangle. (U.L.)

5†. In tunnelling a hill for an aqueduct, two stations P and Q on the centre line are invisible from each other, but are both visible from two mutually intervisible stations R and S, which lie to the north of PQ when $PQRS$ are taken in clockwise order.

Calculate the values of the angles QPR and PQS from the following observed angles :

PSR,	$102° \ 24'$.	QSR,	$42° \ 36'$.
PRS,	$36° \ 48'$.	QRS,	$110° \ 12'$.

(U.L.)

[*Hint.* If r and l are respectively the right and left products of the sines of the known angles, and $m = \alpha + \beta$, where $\alpha = QPR$ and $\beta = PQS$, then

$$\cot \alpha = -\frac{r + l \cos m}{l \sin m}.$$

($\alpha = 52° \ 56' \ 27{\cdot}6''$; $\beta = 26° \ 27' \ 32{\cdot}4''$)]

6†. The essential part of the triangulation for an alpine tunnel consists of a quadrilateral $ABCD$ with these stations taken in clockwise order. A and C are not intervisible and, although B and D are intervisible, BD can be calculated only with respect to the adjacent sides.

Show that the angle $A_2 = BAC$ must be determined by a quadratic of the form :

giving the values of a and b in terms of the observed angles. (U.L.)

7†. Give an account of the methods that have been used in setting-out the centre lines of long tunnels (i) when shafts can be sunk, and (ii) when shafts cannot be sunk. (U.L.)

8†. A straight tunnel is to be driven through a mountain which rises to a maximum height of about 1200 ft. above formation level. The length is nearly $3\frac{1}{2}$ miles, falling from east to west on a uniform gradient of 1 in 120.

The direction of the centre line can be ranged out over the surface, passing from high ground in either direction to valleys near the entrances to the tunnel.

Describe concisely how you would conduct the setting-out work, in order that operations may proceed simultaneously from both ends.

9*. In the excavation for a tunnel, two adits (or access tunnels) are driven from points P and Q in directions perpendicular to the centre line of the tunnel, meeting the latter respectively at R and S, from which points the tunnel is driven simultaneously in directions 48° 15′ N.E. and S.W. to pass through a point O.

Calculate the lengths PR, QS, of the adits, also the length on the centre line, the following total co-ordinates being given :

P :	2686·4 N.,	1792·8 E.,
Q :	4396·8 N.,	3224·2 E.,
O :	3460·6 N.,	2432·6 E.

(U.L.)

[PR, 151·67 ft., QS, 171·67 ft., RS, 2206·47 ft.]

10. Describe how the centre line of a tunnel can be set out so that work can be commenced at both ends. Your description may refer to a mountain tunnel or to a sub-river tunnel, but you are to assume that neither entrance can be seen from the other. Particular attention is to be paid to levels, to direction, and to checking the work. (I.C.E.)

11. Two points on opposite sides of a mountain and about 2 miles apart have to be connected with a straight tunnel at a uniform grade.

No third point can be found from which the other two points are visible.

Describe in outline the preliminary survey operations that are required before the centre line and grade pegs can be set out, and describe in detail the procedure to be followed in the setting out. (U.C.T.)

ARTICLE 8 : UNDERGROUND SURVEYS

Opportunity will be taken at this juncture of introducing one or two terms that arise in the present connection, though the engineer engaged on underground work should have considerable knowledge of applied geology.

Outcrop, dip, and strike. An **outcrop** or **bassett** is a rock face exposed by the denudation of the material overlying a site. A study of outcrops over a site is of prime importance. When the overlying rocks are covered with alluvial deposits, bore-holes must be sunk in order to obtain the necessary information. If, however, the overlying strata are of tabular

or plane formation, the data obtained from bore-holes or outcrops are reliable, but the information thus obtained may be very misleading. Tabular strata are assumed in the given examples, and in order to facilitate the work an introductory note will be given in respect to lines in plane strata.

The position of any plane in space is determined by two lines, or the equivalent " three points ", which should not be in the same straight line. In practice, two fiducial lines in the plane are required : a horizontal line termed the **strike**, and the other, the line of greatest inclination, or **dip**, the lines being mutually perpendicular in plan.

The problems which arise in this connection may be solved (*a*) graphically, or (*b*) analytically ; and in the latter process, the following relations will facilitate the work, as also in cognate examples.

Sloping plane surfaces. Let OO, PP, be plane contours, or horizontal lines in the plane OP, x, the horizontal distance between these lines and θ the angle of steepest slope in OP (Fig. 78).

(*a*) If OO be the assumed direction of the meridian, and a line AB of bearing ϕ and inclination ω to the horizontal lie in the plane ; then since the height of P above A (or O) is $x \tan\theta$, it is equal to the height of B above A, or $x \sec(90° - \phi) \tan\omega$; and

$$\tan\omega = \tan\theta \, . \, \sin\phi, \text{ while } \sin\phi = \frac{\tan\omega}{\tan\theta} \text{ (Fig. 79).}$$

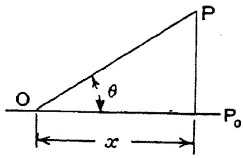


FIG. 78.

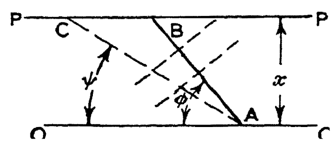


FIG. 79.

(*b*) If, however, the direction ψ is prescribed for a given slope ω along AC, cutting will result if $\psi > \phi$ and filling if $\psi < \phi$ when ω rises from A to C, and the difference in elevation between C and P will be

$$x(\tan\theta - \operatorname{cosec}\psi \tan\omega).$$

(*c*) The lateral slope χ in the stratum at right angles to AB, as shown dotted, will be such that $x \sec(90° + \phi - 90°) \tan\chi = x \tan\theta$; or

$$\tan\chi = \frac{\tan\theta}{\sec\phi},$$

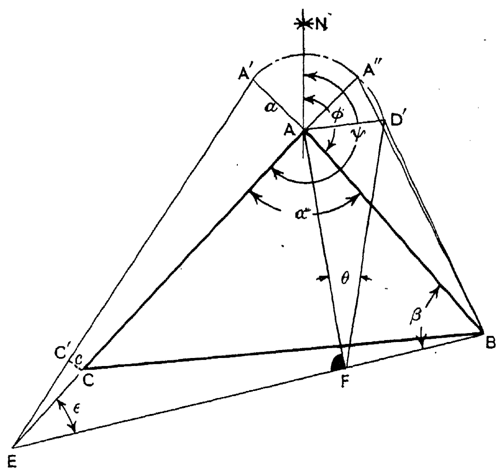
while at right angles to AC, but in the surface,

$$\tan\chi = \frac{\tan\theta}{\sec\iota}$$

Geological three-point problem. Given the borings to three points on a stratum, to determine the dip and strike.

(*a*) *Geometrically.* This is merely the process of determining a horizontal line in a plane surface as given by three points in space, the intersections of the rebatted true lengths meeting the corresponding plans on the datum horizontal line.

Let A, B, and C be the plans of three bore-holes, of which A and B are respectively the highest and lowest points, the levels of A and C with reference to the datum plane through B being a and c accordingly (that is, the indices of the figured plan). Let the bearings of AB and AC be whole circle ϕ and ψ respectively (Fig. 80).



G. 80.

(1) Erect perpendiculars AA'' and AA', CC' to AB and AC respectively, and on the scale of the plan make these equal to the level differences a and c, b being zero. (2) Join $A''B$ and $A'C'$. $A'C'$ will meet the line AC produced in E, giving a second point on the horizontal line BE, which is the direction of the strike. (3) Drop AF perpendicular to EB, and erect AD' perpendicular to AF and equal also to a. Join $D'F$ for the angle of dip θ.

(*b*) *Analytically.* (1) Calculate AE from $\dfrac{a \times AC}{(a-c)}$

(2) Solve the triangle EAB for either ϵ or /

$$\frac{AE-AB}{AE+AB}=\frac{\tan\frac{1}{2}(\beta-\epsilon)}{\tan\frac{1}{2}(\beta+\epsilon)}$$

or
$$\tan\tfrac{1}{2}(\beta-\epsilon)=\frac{AE-AB}{AE+AB}\tan\tfrac{1}{2}(180^\circ-(\psi-\phi)).$$

(3) Calculate $\frac{1}{2}(\beta-\epsilon)$, also $\frac{1}{2}(\beta+\epsilon)=\frac{1}{2}(180^\circ-(\psi-\phi))$. The bearing of the strike will be $180^\circ+\phi-\beta$ or $360^\circ-\psi+\epsilon$, where ϕ and ψ are the given bearings.

(4) Calculate the angle of dip from

$$\tan\theta=\frac{a}{AB\sin\beta}.$$

The strike of the stratum can be set out on the surface of the ground from the computed bearing in the usual way.

Tunnelling through a stratum. The geometrical solution of the problem of the intersection of the centre line of a tunnel with a rock stratum is a simple matter; but the analytical process is somewhat tedious and frequently unwarranted in practice, since the assumption of a plane stratum may or may not be true in any particular case. The sites for the bore-holes should be selected so that the point sought falls within the triangle of the points A, B, and C. Not only does this render the construction more convenient, but it also leads to greater accuracy by limiting the extent of stratum involved to that defined by the bore-holes.

(*a*) *Graphically.* Let A B, and C be the plans of the borings and P a given point on the centre line of the tunnel, as given by co-ordinates or protracted bearings and distances (Fig. 81). Also let a, p, and c be the levels reduced to a datum plane passing through B. (1) Lay down the direction of the centre line PQ on the given bearing, so that PQ meets AB in D and AC in E. (2) At P erect PP' equal to p and perpendicular to PQ; at E erect EE' perpendicular to PQ and equal to $p+PE/g$, where g is the denominator of the gradient, as in $1:g$. Join $P'E'$. (3) At A erect AA' and AA'' perpendicular to AB and AC respectively

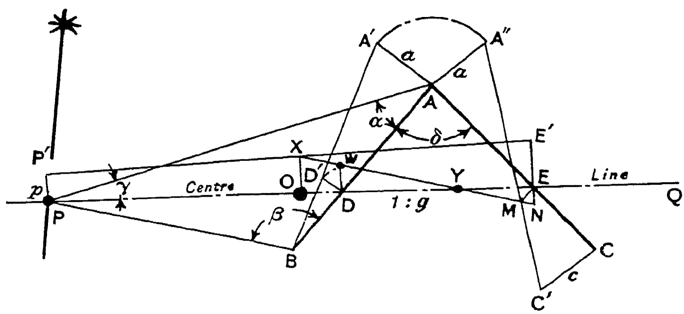


FIG. 81.

and each equal to a. Join $A'B$, giving DD', the height of the perpendicular at D. Make DW equal to DD' and perpendicular to the centre line PQ. (4) At C erect CC' perpendicular to CA and equal to c (here assumed negative, so that C is below B). Join $A''C'$, meeting EM the perpendicular to AC in M, EM being the depth of E *below* B. Erect EN equal to EM and perpendicular to the centre line. Join WN, and produce this line to cut $P'E'$ at X, which is the elevation of the point O in which the centre line intersects the upper surface of the stratum. Drop a perpendicular OX to the centre line PQ, giving O, the plan of the point sought. OX is the depth below B, and PO is the distance of O from P, measured along the centre line of the tunnel.

(*b*) *Analytically*. (1) Calculate the required linear dimensions from the following relations, assuming that AB and AP have been calculated.

$$AD = AP\frac{\sin\gamma}{\sin(\alpha+\gamma)}; \qquad \frac{\sin(\alpha\ [illegible]}{\sin(\alpha+\gamma+\delta)}$$

$$AE = AP\frac{\sin\gamma}{[illegible]}; \quad PD = AP\frac{\sin\alpha}{\sin(\alpha+\gamma)}.$$

(2) Determine (*a*) the level of D on AB with respect to B, and (*b*) the level of D in PQ relative to P. (3) Find the point Y on PQ at the *absolute* level of B. Calculate PO, introducing the following relations :

$$\frac{PO}{g} = \frac{OY}{DY};$$

leading to $$PO = \frac{[illegible]}{DY + g\,.\,DW}.$$

To find the dip of a stratum. *Example**. A, B, and C, forming a right angle at B, are three bore-holes which expose a plane bed of rock, the surface levels involved being given in the following notes :

B.S.	Int. S.	F.S.	Running chainage (feet)	Remarks
13·83				B.M. 225·0
	3·63		0	Ground at A ; 29·20′ above rock.
14·91		1·25	—	
13·07		1·47	—	
	2·15		306	Ground at B ; 15·14′ above rock.
1·29		13·75	—	
	6·65		714	Ground at C ; 21·38′ above rock.
1·65		14·87	—	
	3·21		1224	Ground at A.
		13·40	—	B.M. 225·0.

Reduce the above level notes completely, ascertaining the rock levels at A, B, and C, and determine thence the "dip", or inclination of the steepest line in the stratum, stating its direction relative to AB. (U.L.)

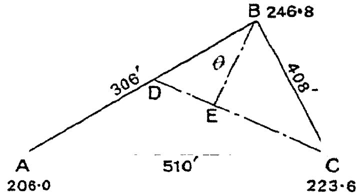


FIG. 82.

From the notes, the reduced surface levels at A, B, and C are 235·20, 261·94, and 244·98 respectively, the stratum levels being 206·0, 246·8, and 223·6 accordingly, as indexed on Fig. 82.

$$B - A = 40{\cdot}8 \; ; \quad B - C = 23{\cdot}2.$$

Assuming D at R.L. of C, then for a horizontal line in the plane CD,

$$\frac{23{\cdot}2}{\;}\, 306 = 174{\cdot}0 \text{ ft.}$$

Also $$\tan\theta = \frac{174}{408} = 0{\cdot}4265 \; ;$$

and $$\theta = 23^\circ\, 6' \; ; \quad \sin\theta = 0{\cdot}3932.$$

Since direction of dip BE is at right angles to CD, AB is at $\theta = 23^\circ\, 6'$ with direction of dip.

Also $$BE = BC \,.\, \sin\theta = 408 \times 0{\cdot}3923 = 160{\cdot}06 \text{ ft.}$$

$$\tan \text{dip} = \frac{23{\cdot}2}{\;} = 0{\cdot}14446 \; ; \; \text{dip} = 8^\circ\, 13'.$$

QUESTIONS ON ARTICLE 8

1††. The following notes refer to the setting-out of a tunnel from a point P to meet the upper plane surface of a rock stratum as determined by bore-holes at A, B, and C, PQ being the centre line of the tunnel which is to be on a rising gradient of 1 in 110.

Line	Distance (feet)	Bearing	Reduced levels
PQ	—	N. 81° 15′ E.	904·2 C.L. of tunnel at P
PA	1940	N. 59° 30′ E.	1184·6 Rock surface at A
PB	1820	S. 87° 20′ E.	894·8 ,, ,, B
AC	1488	S. 49° 30′ E.	704·4 ,, ,, C

(*a*) With the short edges of the paper parallel to the meridian, plot the points P, A, B and C by co-ordinates to a scale of 100 ft. to 1 inch, making P 100 ft. N. and 0 ft. E.

(*b*) Determine the co-ordinates of the point O in which the centre line, having passed through the rock, meets the upper surface of the stratum. Find also the reduced level of O and figure these values on your drawing.

An analytical solution, if preferred, will be accepted. (U.L.)

[Co-ordinates of O, 429·7 N.; 2141·8 E.; R.L., 923·9]

2*. A rectangular building plot 250 ft. × 150 ft. in plan has its long sides running due east, and its surface is plane with the steepest slope of 1 in 8 downwards in a direction of N. 60° W. from the S.E. corner. Neglecting side slopes, calculate the volume of earthwork involved in *excavating* to a foundation level of 27·40, the reduced level of the S.E. corner being 64·60. (U.L.)

[Slopes parallel to long and short sides respectively, 0·1082 and 0·0625, expressed as tangents, and cuts of 37·20; 27·83; 10·15 and 0·78 ft., leading to a volume of 712,125 cu. ft. or 26,375 cu. yds.]

ARTICLE 9: HYDROGRAPHICAL SURVEYING

Hydrographical surveying consists in surveying the coast-line and adjacent waters, harbours, estuaries, rivers, etc., taking soundings, etc., for the purpose of preparing a record or chart which will indicate such features of the locality as affect navigation or constructional operations.

The subject may be divided into (I) marine surveying and (II) river surveying.

(I) **Marine surveying.** An extensive coast survey may be made either (1) *by triangulation and traversing* on the shore, or (2) *from small boats and the ship.*

(1) In an extensive survey, the principal points on the *high-water lines* are determined by triangulating and the sections between these stations (or the primary lines of smaller surveys) are traversed, the lines being run to follow the shore-line approximately, so that bends, etc., may be fixed by offsets.

The foregoing method is either difficult or impossible in the case of the low-water line, which is bare for only a short time. Here the survey is best continued with the sextant, the station of observation being fixed by angles θ and ϕ observed to three visible and charted points A, B, and C on the shore.

When the shore is inaccessible, a base must be measured on the water, usually by fixing its ends by intersections from the shore, the shore-line and high- and low-water marks being then surveyed by sextant angles from each end of the base.

(2) Running surveys are made by determining the ship's position if possible (*a*) by three-point resection, or (*b*) from a base measured by sound signals, a shore party and a ship's party firing guns alternately.

Soundings. Soundings are depths measured below the surface to the bottom of a river, lake, bay, etc., by means of sounding lines; chains and rods are frequently used in engineering surveys. Soundings are to hydrography what reduced levels are to topography: they provide the data by which the submarine surface is represented conventionally on a plan or chart. The datum to which soundings are referred is the mean low-water of ordinary spring tides (L.W.O.S.T.). This datum will serve also for engineering surveys when it is sensibly horizontal; but on extensive surveys of the coast, or in tidal waters, the low-water level varies so considerably that a fixed datum must be employed. Thus levels and soundings are commonly referred to the nearest Ordnance bench marks, and the working datum must be suitably established with additional bench marks. Engineering plans are usually on the 6-inch and the 25-inch scales, soundings and contours being expressed in feet in preference to the fathoms characteristic of charts.

Tide gauges are erected at suitable points to enable the surveyor to determine the exact level of the water surface during the whole time that the soundings are being taken. In quiet waters a divided rod or pole, showing feet and tenths, is used, the zero being referred to the datum by spirit levelling. In rough waters a perforated pipe with a vertical graduated float is used, a mechanism being sometimes added to trace the fluctuations of the tide on a rotating cylinder. Sometimes the entire rise and fall of the tide may be ascertained from a single gauge, but frequently a succession of gauges is installed, forming a series of steps, each zero being fixed by careful levelling.

The gauges are read every five, ten, or fifteen minutes according to circumstances and the nature of the work, while in tidal observations they are often read every ten minutes in the hour before and the hour after both high- and low-water, and every half-hour during the remainder of the twenty-four hours. Since, however, the gauges are read at regular intervals, the exact reading of the gauge must be found by interpolating between the readings. Hence the necessity of always recording the time of taking a sounding. Whenever no regular gauge readings have been observed, it is possible to reduce the soundings by algebraical substitution in the following formula:

$$h = h_0 + y \cos 180^\circ\, t/T,$$

where h is the height of sea-level above datum at the time of a given sounding, h_0 the height of mean sea-level above datum, y the rise of the tide above mean sea-level, t the time interval between time of high-water and time of sounding, and T the time interval between high- and low-water.

Fixing soundings. The positions of soundings may be fixed in various ways as follows :

(1) From the shore by means of the intersecting angles observed simultaneously with theodolites stationed at the extremities of a measured land base ; (2) on the prolongations of perpendiculars from the shore, the sounding boat keeping to a perpendicular while the *alternate* angle from the other perpendicular is (*a*) observed on the land with a theodolite, or (*b*) from the boat with a sextant or compass ; and (3) by trilinear coördinates (or three-point resection) from the boat by means of the sextant or compass.

Subsidiary methods are also used as a means of interpolation, time intervals between main soundings being observed, sometimes with the patent log, or by anchor lines paid out for regular distances.

(II) **River surveying.** Soundings in rivers and narrow waters are often taken on zig-zag lines from shore to shore in equal intervals of time, the work being carried out from motor launches with the patent log towed astern while its indicator is fixed to the gunwale. Frozen rivers allow of the use of the more precise methods of land surveying.

Surveys of rivers are executed in various ways ; sometimes by combining soundings and survey, and sometimes by running soundings between land stations. A combined triangulation method consists of the following, four boats being used, each provided with log, sextant, and compass : A base is measured across the mouth of a river, and two boats proceed from the ends of this base, surveying and sounding along opposite shore lines to the next stations, while the other two boats observe angles and sound across to those stations. On reaching these, the latter boats proceed along the next run of shore-line, while the former boats proceed along the diagonals, and so on alternately until the required stretch of river has been surveyed.

The principles of trilinear co-ordinates are usually associated with hydrography, although the three-point problem occurs in plane tabling and other connections. The solution may be (1) *mechanical* by means of the **station pointer**, a three-armed protractor, the use of which corresponds with the tracing-paper trammel in plane tabling ; (2) *graphical*, the angles θ and ϕ otherwise observed with the sextant being constructed goniographically in the case of the plane table ; and (3) *analytical*, in more precise work, particularly with the theodolite (see p. 164). Trial methods, eliminating the "triangle of error", are sometimes used in plane tabling.

The problem is insoluble when the three observed points A, B, and C and the point of observation P lie on the circumference of a circle.

Three-point problem. Given θ and ϕ, the horizontal angles subtended at a sounding point P by three charted and visible points A, B, and C, to determine the position of P.

Graphical solutions.

(*a*) *By the intersection of a straight line and circle.* (1) At A draw AE so that the angle $CAE = \phi$; and at C, draw CE so that the angle $ACE = \theta$. (2) Construct a circle through A, E, and C, and draw a line through B and E, cutting the circle in the required point P.

From Fig. 83 it will be seen that the angles APE and CPE are respectively equal to θ and ϕ, the angles in the same segment of a circle being equal.

(*b*) *By two intersecting circles.* (1) Construct angles at A and B equal to $90° - \theta$, and at B and C equal to $90° - \phi$, giving the points O_1 and O_2. (2) Describe circles with centres O_1 and O_2 respectively through A and B and B and C, giving the required point P at their intersection. Join

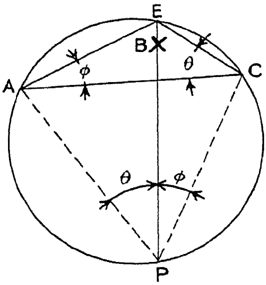


FIG. 83.

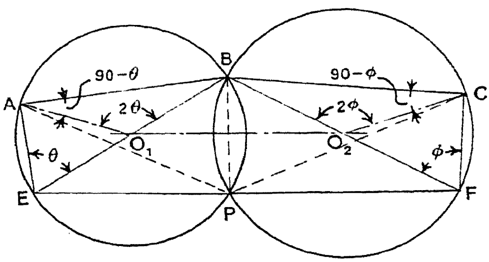


FIG. 84.

AP, BP, and CP. Then since in any circle the angle subtended at the circumference is one half the angle subtended at the centre, the angle $APB = \frac{1}{2}AO_1B = \theta$, and the angle $CPB = \frac{1}{2}CO_2B = \phi$.

Now if the diameters BO_1 and BO_2 be produced to meet the circles in E and F respectively, and AE, CF, and EF be joined, the angles BAE and BCF will be right angles, while by the first theorem the angles AEB and CFB will be respectively equal to θ and ϕ, P being the foot of the perpendicular let fall on EF from B.

These corollaries are the basis of the author's method of solving the three-point problem on the plane table, the circles being avoided by the construction of right angles at A and C.

Analytical solutions. These will be introduced through the medium of the following numerical example.

Example†. The computed sides of a triangle ABC with stations in clockwise order are AB, 5650 ft. ; BC, 6860 ft. ; and CA, 9445 ft. Outside this triangle (and nearer to AC) a station P is established and its position is to be found by three-point resection on A, B, and C, the angles APB and BPC being respectively 42° 35′ and 54° 20′.

Determine the distances PA and PC. (U.L., Cart.)

In Fig. 84, let $AB = x$, $BC = y$, with $APB = \theta$ and $BPC = \phi$, ABC being ψ.

Also let $\delta = 360° - \theta - \phi - \psi = \alpha + \gamma$, the sum of the unknown angles at A and C respectively.

$$\text{Now} \quad PB = \frac{x \sin \alpha}{\sin \theta} = \frac{y \sin \gamma}{\sin \phi}, \quad \text{and} \quad \sin \gamma = \frac{x \sin \alpha \, . \sin \phi}{y \sin \theta} \quad \text{............(1)}$$

Substituting for $\gamma = \delta - \alpha$, $\sin \gamma = \sin \delta \cos \alpha - \cos \delta \sin \alpha$, ...(2)

$$\text{and} \quad \frac{x}{y} \frac{\sin \alpha \sin \phi}{\sin \theta} = \sin \delta \cos \alpha - \cos \delta \sin \alpha. \quad \text{......................(3)}$$

$$\text{Whence} \quad \frac{x}{y} \frac{\sin \phi}{\sin \theta} = \sin \delta \cot \alpha - \cos \delta ;$$

$$\text{and} \quad \cot \alpha = \cot \delta \left(\frac{x \sin \phi}{y \sin \theta} \cos \delta + 1 \right). \quad \text{......................(4)}$$

First calculating ψ with $AC = z$,

$$z^2 = x^2 + y^2$$

$$\text{or} \quad \cos \psi = -\frac{z^2 - x^2 - y^2}{2xy} = -\frac{(9445)^2 - (5650)^2 - (6860)^2}{2(5650)(6860)} = -0{\cdot}1328980$$

$$= \cos(180° - 82° \, 22' \, 14'')$$

$$\psi = 97° \, 37' \, 46'', \quad \text{with} \quad \delta = 165° \, 27' \, 14''.$$

By Eq. (4),

`7′ 14″) 5650 sin 54° 20′

$$= -3{\cdot}98154 + 3{\cdot}93594 = -0{\cdot}04560 = \cot 92° \, 36' \, 39''.$$

$$' = 180° - 92° \, 36' \, 39'' - 42° \, 35'$$

$$= 44° \, 48' \, 21''$$

$$= 5884{\cdot}17 \text{ ft.}$$

$$\text{Also} \quad PC = \frac{y \sin CBP}{\sin \psi}, \quad \text{where } CBP = 180° - 72° \, 50' \, 35'' - 54° \, 20'$$

$$= 52° \, 49' \, 25'',$$

$$\text{checking } \psi = 97° \, 37' \, 46''$$

$$= 6730{\cdot}03 \text{ ft.}$$

Plane table resection. Mention may be made here of the graphical solutions of the foregoing problem, in which the visible points A, B, and C will be plotted as a, b, and c on the board, where p will represent P, the station occupied by the table. In this connection the angle θ in Fig. 83 would be set out by sighting along ca to B, and, with the board clamped thus, drawing a ray from c in the direction of B; likewise, for the angle ϕ, the rays drawn towards B from c and a intersecting at e on the board. Orientation would follow by sighting along be to B, and with the board

thus clamped, sighting through a to A, the ray intersecting be produced in p. A check on p thus determined would follow from a ray drawn through c towards C.

The angles aeb and bfc are constructed in a similar manner in the author's method, where the perpendiculars at a and c are drawn previously (Fig. 84).

Two-point problem. Given a and b, the plotted positions of A and B, two points visible from P, the point selected for a plane table station; to plot p, the position of P on the board.

A magnetic meridian drawn on the map is sufficient only for *setting* the board by means of the compass, and, although p might be thus determined by simple resection, the process is admissible only to rough work or very small scales.

The following method, which may be entirely independent of the compass, may be detailed with reference to a quadrilateral $ABQP$, drawn with the stations in clockwise order with AB lying roughly west to east and north of PQ. The intersections on A and B from stations at P and Q will give $ab'qp$ with ab' (not ab) parallel to AB.

(1) Select a fourth point Q so that the intersections from P and Q will be satisfactory. Occupy Q first, and orient the board by estimation (aided by the compass, if available). Centre the alidade on a and b, and, sighting A and B accordingly, draw rays intersecting at q. Centre the alidade on q, sight P, and draw a ray qs. (2) Occupy P, and orient the board by sighting Q with the alidade along sq. Centre the alidade on a, sight A, and draw a ray, intersecting sq in p. Centre on p, sight B, and draw a ray, intersecting qb in b'. (3) Fix a picket R by sighting along ab'; then with the alidade along ab, turn the board until R is seen in line. Clamp the board with ab thus parallel to AB, and finally fix p_0 by resection, sighting through a to A and through b to B.

QUESTIONS ON ARTICLE 9

1†. Derive a solution to the Snellius' three-point problem.

2†. In a triangulation survey it becomes necessary to incorporate a station S not in the original net, and its position is determined by angular observations on three visible stations P, Q, and R, the total co-ordinates of which are appended, with the two horizontal angles observed from S.

Station	Latitude	Departure	Angle
P	+ 18,400	+ 72,800	52° 12′ 20″
Q	+ 18,400	+ 94,600	68° 30′ 15″
R	+ 2,200	+ 107,400	

Determine *analytically* the co-ordinates of the station S. (U.L.)

[Lat. − 1120; Dep. + 91,934.]

3*. Discuss the problem of Three-Point Resection in its relation to plane-table surveying; and explain clearly with a sketch the geometrical basis of a graphical solution. (U.L.)

4*. A, B and C are three visible and charted points in a hydrographical survey and θ and ϕ are the angles observed with a sextant between A and B and B and C respectively from a sounding boat at P.

Prove the geometrical process of plotting the position of P by means of two intersecting circles, and state when the solution fails. (U.L.)

5††. The following notes refer to a triangle ABC in which a beacon Q was sighted from A and C in mistake for the signal at B.

After the mistake was discovered the distance BQ was measured, being 164·5 ft. eastwards of B.

Station	Total co-ordinates	
	Latitude	Departure
A	+ 726·25	+ 844·65
B		
C	+ 802·50	+ 1636·45
D	+ 1433·52	+ 1237·67

Determine the co-ordinates of B, given that the observed angle ABC is 62° 44′ 10″. (U.L.)

[This is not a case of resection. The locus of B is a circle through A and C. Co-ords. of B: Lat. 1392·50; Dep. 1078·36.]

6*. In a photographic survey it is deemed essential to occupy a random point P, not in the triangulation, as a camera station; and P is fixed in position by angles θ and ϕ observed from P to three visible triangulation stations A, B, and C, no linear measurements being made. Describe with reference to sketches how you would plot the additional station P by means of a graphical construction (as distinct from a mechanical device). (U.L.)

7*. In a harbour development scheme at the mouth of a tidal river, it has been found necessary to take soundings in order to buoy the navigation channel.

Explain clearly how you would determine the levels of points on the river bed and fix the positions of the soundings:

(*a*) by use of sextant in a boat;
(*b*) by use of theodolite on the shore. (U.L.)

8†. In connection with harbour developments at the mouth of a tidal river, it is found necessary to take soundings in order to buoy the navigation channel.

Clearly explain how you would fix the position and levels of points on the river bed:

(*a*) by use of a sextant in a boat;
(*b*) picking up the points of soundings from the shore. (U.L.)

9*. In connection with a harbour works, soundings were taken along the prolongation of a line PQ, as determined by two flag poles on the shore, and the position of the sounding boat in PQ produced was fixed by the following angles, which were observed with a theodolite stationed on the shore at O, on a perpendicular to PQ at Q, 175 ft. to the right, the A vernier reading zero for sights taken along the parallel to PQ through O.

A tide gauge with its zero at Low Water of Ordinary Spring Tides (L.W.O.S.T.) was read concurrently with the soundings :

Sounding No.	1	2	3	4	5	6	7	8
Observed angle - -	290°	304°	317°	325°	330°	334°	337°	340°
Depth of sounding (ft.) -	4·3	6·7	9·0	11·4	13·9	16·1	18·5	21·0
Tide gauge (ft.) - -	5·2	5·2	5·2	5·3	5·3	5·3	5·4	5·4
Distance from Q (ft.)								
Depth below L.W.O.S.T.								

Duplicate the last two lines of the foregoing table, and reduce in these the required information. (U.L.)

[64, 118, 188, 250, 304, 359, 412, 481 ft.
− 0·9, 1·5, 3·8, 6·1, 8·6, 10·8, 13·1, 15·6 ft.]

10*. You are required to determine the 1 fathom contours of the bed of a harbour prior to extensions.

Describe how you would proceed with the following equipment and personnel at your disposal :

Sextant, theodolite, dumpy level and staff, two tide gauges, sounding line, watches, and a boat ; 1 assistant, 2 handymen, and 2 boatmen. (U.L.)

11*. Describe the method by which the base of a marine survey may be measured by sound signals.

The line between two ships A and B has a magnetic bearing of N. 13° W., and the direction of the wind is from 42° E. of N. The observed time intervals between the flash seen at A and the report heard at B, and *vice versa*, are $9\frac{1}{5}$ and $9\frac{3}{5}$ sec. respectively, the mean temperature being 46° F.

Determine the distance between the ships, given that sound travels 1090 ft. per sec. at 32° F., with a gain of $1\frac{1}{2}$ ft. per degree F. (U.L.)

[$D = (1090 + a(\text{F.}^\circ - 32^\circ))T\cos\theta$, where $T = \dfrac{2tt_1}{t+t}$ is the mean time interval (sec.) and t and t_1, the intervals (sec.) as observed respectively on shore and aboard ship, a the gain per sec. (usually $\frac{11}{7}$ ft.) per degree above 32° F., and θ the direction the wind makes with the direction of sound. Dist. 5988 ft.]

12. A large town is situated at the mouth of a river flowing from west to east. It is desired to carry out a survey of the channel for a few miles seawards of the river mouth. On one side of the river the land runs north-eastwards beyond the mouth for several miles. Describe a simple and accurate method of locating the position of soundings, using one sextant only. It may be assumed that an accurate map of the town and its surroundings exists.

What other observations besides soundings and sextant measurements must be taken? (I.C.E.)

13. The following information is required with regard to the entrance to a sea lock:

(1) Shore line survey and soundings for location of navigation beacons and possible channel improvements.

(2) Speed and direction of currents at various stages of tide.

The passage to be surveyed is about half a mile wide and a mile long. For the greater part the coast line is very rough and the land slopes steeply and is thickly wooded, but at the ends there are stretches where good chaining conditions occur.

Discuss methods of carrying out these operations, illustrating them by sketches. (U.G.)

14. In the course of a marine survey an observer takes sextant angles AXB, BXC subtended at the boat X by the points, A, B, and C on shore. If A, B, and C are shown on his chart, describe how he may plot the point X (*a*) instrumentally, (*b*) by graphical construction.

Explain what are meant by good and bad fixes in the location of points such as X. (I.C.E.)

15. Explain carefully how you would make a survey of the bed, banks and currents of a tidal estuary. The width of the estuary is about $\frac{3}{4}$ mile, and the reach to be surveyed is nearly straight and about 1 mile in length. The banks are low and the maximum depth of water is 30 feet. (I.C.E.)

16. Describe and sketch the following items of equipment used in marine surveying: a sounding rod, a water telescope, a float tide gauge, a tidal current float. (U.D.)

ARTICLE 10: LATITUDES AND DEPARTURES

A brief recapitulation of the applications of this rectangular system of co-ordinates appears to be desirable since the scope of the method is seldom appreciated, comprehending, as it does, the following operations: (1) plotting surveys; (2) adjusting surveys; (3) supplying omitted measurements; (4) calculating areas; (5) parting land; and (6) determining obstructed distances.

The **latitude** λ and **departure** δ of a line of length s are respectively $s.\cos\beta$ and $s.\sin\beta$, where β is the reduced (or quadrant) bearing, in which the initial letter gives the sign of the latitude (N + ; S −) and the terminal letter the sign of the departure (E + ; W −), the plus values being styled **northings** and **eastings** and the minus values **southings** and

(1) **Plotting surveys.** Not only is the method used in plotting traverses, but also triangulation nets in certain surveys. In this connection an origin is assumed, and total co-ordinates are found by adding algebraically therefrom the individual or consecutive co-ordinates of the several courses, as calculated from $s \, . \cos \beta$ and $s \, . \sin \beta$. Usually the origin is placed at the most westerly station, which is identified in traverses by observing when the terminal letter of the successive bearings changes from west to east. Frequently, however, a rough plot may be advisable. Also it should be noted that in large-scale maps the north, true or magnetic, may not be parallel to an edge of the paper in order that an approach road, etc., may be placed in the most appropriate position. Hence it may be necessary to assume an arbitrary meridian parallel to the edges of the paper, and to change all the bearings by a definite angle before calculating the latitudes and departures. Otherwise the tedious process of re-calculation may be necessary. In many surveys a network of graticules, or grid, is used, the sides of the squares varying from 20 ft. to 200 ft. according to the objects and scale.

(2) **Adjusting surveys.** The true error of closure of a traverse is given by $E = \sqrt{E_l^2 + E_d^2}$, where E_l and E_d are respectively the errors from zero in the algebraical sums of the latitudes and departures. This subject will be treated at length in a later section.

(3) **Omitted measurements.** This artifice is also based upon the fact that in a closed traverse the algebraical sums of the latitudes and departures should be zero; and it follows that any two quantities can be supplied; namely, two bearings, two sides, or a side and a bearing. Nor need the missing quantities apply to one side or adjacent sides, for in other cases a closing line is made to complete a figure with the unaffected sides, and with this closing line and the two elements of the affected sides, a triangle is formed and is solved by plane trigonometry. Simultaneous equations appear to afford a plausible means of solution, but, lacking method, almost invariably lead to confusion.

(4) **Areas.** When the co-ordinates of a traverse have been calculated, it is a simple matter to use these in determining the area of the skeleton, which occasionally may represent an area with straight fences.

The method of "double longitudes" is possibly the most convenient, the doubling avoiding the "half" in the areas of triangles and trapezoids. The double longitude of any side is the double longitude of the preceding side *plus* the departure of the preceding side *plus* the departure of the side itself, the double longitudes of the first and last sides being merely their departures. Each double longitude is multiplied by the latitude of the same side, north latitudes giving North Products (N.P.) and south latitudes South Products (S.P.). The area $A = \frac{1}{2}(\Sigma\text{N.P.} \sim \Sigma\text{S.P.})$ (p. 171).

The method is exceedingly useful in replacing a crooked boundary by a straight one so as to contain the same area.

(5) **Partition of land.** This problem may conveniently follow the calculation of an area by the foregoing method, and in this connection and in rectifying crooked boundaries, the area A must be known. There are two general cases : (a) Partition by a line through a given point, and (b) partition by a line in a given direction. Usually a ratio k of the area A is specified, and a line is drawn across the survey, cutting off what appears to approximate to the area kA. This dividing line gives closure between the parts of the area, and from the relevant data the extremities of the line are fixed, though in Case b the solution is largely by trial.

(6) **Obstructions.** In measuring or interpolating points in the obstructed distance between two stations A and B which are not intervisible, a zig-zag traverse is run between A and B, the direction of the first line of this traverse being assumed as the reference meridian, and the angle between AB and this assumed meridian is found from $\tan \alpha = \Sigma\delta/\Sigma\lambda$, where λ and δ are respectively the latitudes and departures. If then the direction of the reference meridian is changed to that of AB, the total departure will be zero on closing the original traverse (or a more convenient one) on any point in AB.

Example†. The following bearings and distances were recorded in running a theodolite traverse in the counterclockwise direction, the bearing of CD and the length of DE having been omitted :

AB due North, 1020 ft. ; BC, N. 22° 15′ W., 576 ft. ; CD, ... , 2604 ft. ; DE, S. 58° 10′ E., ... ; and EA, N. 36° 30′ E., 720 ft.

Determine the omitted measurements on the assumption that the recorded measurements are uniformly precise. (U.L.)

Summing algebraically the known latitudes and departures as follows :

Line	Latitude	Departure
1. AB	+1020·00	0·00
2. BC	+ 533·11	−218·10
3. CD	—	—
4. DE	—	—
5. EA	+ 578·78	+428·27
	+2131·89	−210·17

Wherefore Lat. and Dep. of temporary closing line CE, giving length

$$CE =$$

$$=2142·5 \text{ ft.}$$

Bearing of CE;

$$\tan\beta = \frac{-210\cdot17}{-2131\cdot89} = 0\cdot098584.$$

$\beta = 5°\ 37\cdot8'$ S.W.

$$\sin D = 2142\cdot5 \frac{\cdot3}{2604} = 0\cdot73820;$$

$D = 47°\ 34'\ 42''$.

Bearing γ of CD, S. 74° 15·3′ W.

Length DE

$$= \frac{\sin 68°\ 37\cdot5'}{\sin 63°\ 47\cdot8'}$$

$= 2702\cdot6$ ft.

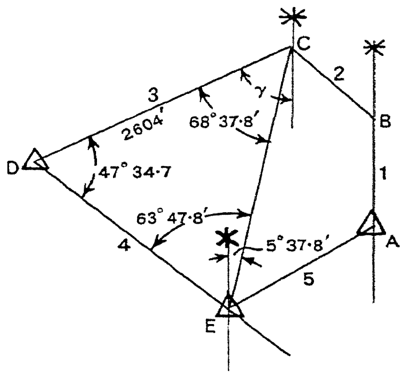


FIG. 85.

Example†. The following notes were recorded in a theodolite and chain traverse :

AB, S. 25° 00′ E., 3·88 chs. ; BC, N. 90° 00′ E., 9·86 chs. ; CD, N. 66° 00′ W., 6·24 chs. ; DE, N. 31° 36′ W., 4·47 chs. ; EA, S. 50° 51′ W. 4·50 chs.

(a) Record these notes on an appropriate tabular form, and calculate the latitudes and departures of the several courses.

(b) Adjust the traverse to close by Bowditch's method, assuming errors in proportion to the lengths of the sides.

(c) Calculate the acreage of the traverse, using the corrected co-ordinates.

In order to avoid an inset the solution is abbreviated as follows :

Line	AB	BC	CD	DE	EA		
Obs. Lat. -	−351·65	00·00	+253·80	+380·72	−284·11 lks. ;	$E_l =$	−1·24
Obs. Dep. -	+163·98	+986·00	−570·05	−234·22	−348·97 ,,	$E_d =$	−3·26
Correction (l)	− 0·17	+ 0·42	+ 0·27	+ 0·19	− 0·19 ,,	Sum	1·24
,, (d)	+ 0·44	+ 1·11	− 0·70	− 0·50	− 0·51 ,,	,,	3·26
Corr'd Lat. -	−351·48	+ 0·42	+254·07	+380·91	−283·92 ,,	,,	0·00
,, Dep. -	+164·42	+987·11	−569·35	−233·72	−348·46 ,,	,,	0·00
Double Longs.	164·42	1315·95	1733·71	930·64	348·46 ,,	,,	
N. Products -	—	553	440484	354490	—	Sum	764962
S. Products -	57787	—	—	—	98935	,,	156722
S.						2)	638805

Area $A = 3\cdot19403 = 31\cdot9403$ sq. chs. = sq. lks. 319403

Example†. Describe the process of dividing the area of a traverse survey into specified parts :

(a) By a line through a given point.

(b) By a line in a given direction.

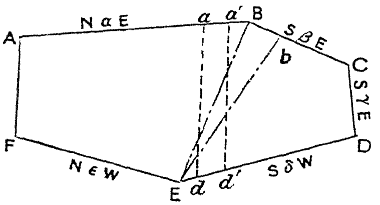


FIG. 86.

Case (*a*). Let $ABCDEF$ be a farm of area A, which is to be divided by a line through E, cutting off a portion kA, where k is a specified percentage, say, 40 per cent (Fig. 86).

It will be seen by inspection that $ABEF$ is somewhat less than $(1-k)A$, or 60 per cent, and the line of partition must have some position, such as Eb, where b is to be determined.

(1) Calculate the whole area A. (2) Calculate the area $ABEF$, and find the correctional area to be applied in the triangle BEb. (3) Find the direction and length of EB as for a missing line; that is, if x and y are values making the respective sums of the latitudes and departures zero, $\tan\theta = y/x$, where θ is the required bearing, and $BE = \sqrt{y^2 + x^2}$. Also the angle bBE will be known from the difference of the bearings β and θ of BC and BE respectively. Hence the corrective area:

$$\text{or } bB = \frac{2a}{EB \,.\, \sin bBE}.$$

Case (*b*). Let it be required that partition be made by a line parallel to AF.

(1) Assume the position of the dividing line, ad, say, fixing a definite distance for Aa. (2) Find the lengths ad, dE, as omitted measurements, thus:

$$ad \,.\, \cos\phi + dE \cos\delta = X\ ; \text{ and } ad \,.\, \sin\phi + dE \sin\delta = Y.$$

From these find ad and dE, ϕ and δ being known.

(3) Calculate the area $AadEF$, preferably by the method of double longitudes (p. 169). (4) Since it is not likely that the specified ratio k will result, move ad parallel to itself to $a'd'$. The solution must be by trial.

Example†. The following notes show the courses and co-ordinates of a survey with straight fences, the area being 9·7396 acres.

Divide the area into two plots of equal area by means of a line perpendicular to AB, locating the ends of the dividing line.

Line	Length (ft.)	Bearing	Latitude	Departure
AB	986·0	N. 90° 0′ E.	00·00	+986·00
BC	388·0	N. 25° 0′ W.	+351·30	−164·20
CD	447·0	N. 31° 36′ W.	+380·40	−234·00
DE	613·5	S. 50° 51′ W.	−387·30	−457·80
EA	368·0	S. 20° 41′ W.	−344·40	−130·00

(U.L.)

Since division will be by some such line as pq parallel to the meridian (Fig, 87),

$$Ap \cos 90° + pq \cos 0° - qE \cos 50° 51' + \text{lat. } EA = 0 ;$$

$$pq = qE \cos \delta + 344{\cdot}4. \quad \ldots\ldots(1)$$

$$Ap - qE \sin 50° 51' + \text{dep. } EA = 0 ;$$

$$Ap = qE \sin \delta + 130{\cdot}0. \quad \ldots\ldots(2)$$

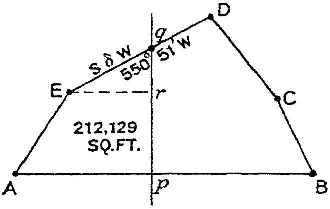


FIG. 87.

Area $\qquad A = \frac{1}{2}Ap \,.\, pr + \frac{1}{2}pq \,.\, qE \sin \delta,$

or $\qquad 2A = \{Ap(pq - qE \cos \delta) + pq \,.\, qE \sin \delta)\}. \qquad .(3)$

Substituting for pq and Ap from (1) and (2),

$$2A = (qE)^2 \sin \delta \cos \delta + 688{\cdot}8\, qE \sin \delta + 130 \times 344{\cdot}4 \; ; \; A = 212{,}129 \text{ sq. ft.}$$

$$(qE + 545{\cdot}51)^2 = 1072659 \; ; \; qE = 490{\cdot}17 \text{ ft.}$$

Whence from (2), $Ap = 490{\cdot}17 \times 0{\cdot}77550 + 130 = 510{\cdot}13$ ft.

$$Ap = 510{\cdot}13', \text{ N. } 90° \, 0' \text{ E. } ; \; qE = 490{\cdot}17', \text{ S. } 50° \, 51' \text{ W.}$$

Example†. In a constructional survey two stations P and Q are not intervisible on account of various obstructions, and it is required to peg out the line, particularly near c, of the following traverse. Accordingly a zig-zag course $PbcdQ$ is run, a reference meridian being assumed along Pb.

Pb: 227·7 ft., 0° 0′ ; bc: 224·2 ft., N. 75° 47′ W. ; cd: 242·1 ft., N. 23° 11′ W. ; and dQ: 149·9 ft., N. 85° 16′ W.

State the direction in which the line is to be ranged out from P, and give the notes relative to interpolating a point x in PQ near c. (U.L.)

Calculating the co-ordinates with reference to a meridian along Pb (Fig. 88):

Line :	Pb	bc	cd	dQ
Lat :	+227·7	+ 55·6	+222·6	+ 12·37 ft.
Dep. :	0·0	−217·5	− 95·2	−149·5 ft.

$$\tan \alpha = \frac{\Sigma \text{ Dep.}}{\Sigma \text{ Lat.}} = \frac{-462{\cdot}2}{+518{\cdot}3} = -0{\cdot}891 \; ; \; \alpha = \text{N. } 41° \, 42' \text{ W.}$$

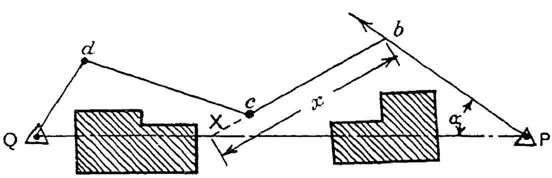


FIG. 88.

Changing the meridian to the direction of PQ, Dep. $Pb = +151{\cdot}4$. Assuming a point in direction bc on PQ at a distance x from b,

$$151{\cdot}4 - x \,.\, \sin 34° \, 5' = 0 \,;\; x = \frac{151{\cdot}4}{0{\cdot}5604} = 270{\cdot}1 \text{ ft.}$$

QUESTIONS ON ARTICLE 10

1†. The following courses were recorded in running a traverse in the counterclockwise direction, the bearing of CD and length of EF having been omitted :

AB, N. 0° 0′ E., 1066 ft. ; BC, N. 22° 15′ W., 576 ft. ; CD, ... ; 2245 ft. ; DE, S. 64° 36′ E., 2353 ft. ; EF, N. 82° 5′ W., ... ; FA, N. 36° 30′ E., 720 ft.

(a) Determine the omitted measurements on the assumption that the recorded measurements are uniformly precise.

(b) Plot the traverse lines by co-ordinates with reference to an origin at the most westerly station, using a scale of 200 ft. to 1 inch. (U.L.)

[Bearing of CD, S. 56° 42·3′ W. ; length of EF, 463·65 ft.
Additional co-ordinates of CD : Lat., 1231·83 ft., Dep., 1875·64 ft.
,, ,, ,, EF : ,, 63·86 ,, ,, 459·23 ,,]

2*. Explain with reference to a five-sided traverse skeleton the process of determining areas by the method of latitudes and double longitudes.

Indicate with specimen notes how the co-ordinate method may be used in acertaining areas alongside boundaries from the field notes. (U.L.)

3*. The following corrected latitudes and departures were calculated for a six-sided traverse in which A was the most westerly station.

Line	Latitude	Departure
AB	+ 720	+ 430
BC	+ 90	+ 810
CD	− 1220	+ 140
DE	− 360	− 760
EF	+ 320	− 340
FA	+ 450	− 280

Calculate the area of the traverse, recording your data in tabular form.
[1,521,250 sq. ft. = 34·923 acres.] (U.L.)

4*. The following notes refer to a theodolite and chain traverse $ABCD$ of a site which is to be divided into two equal areas by a line parallel to BC :

AB, 1240 ft., N. 8° 30′ E. ; BC, 920 ft., N. 89° 50′ E. ; CD, 1,228 ft., S. 3° 30′ E. ; and DA, 1,178 ft., S. 89° 50′ W., the traverse running in the clockwise direction.

Calculate the exact positions of the ends of the partition line with regard to Stations A and D, and state also the common area of each plot, ignoring any errors in the traverse. (U.L.)

[Since only four sides are involved, and AB and DC produced will meet in a point, the use of latitudes and departures is precluded. 581·0 ft. from A on AB ; 575·3 ft. from D on DC.]

5. State and prove the correctness of one of the two methods of determining the area enclosed within the lines of a closed traverse in terms of the latitudes and departures of the various sides. Calculate the area enclosed within the lines of the traverse of which particulars are given below. All lengths are in feet.

Line	Northing	Southing	Easting	Westing
AB - -	—	298	169	—
BC - -	—	151	362	—
CD - -	630	—	383	—
DE - -	301	—	—	560
EA - -	—	482	—	354

(I.C.E.)
[11·07 acres.]

6. Two large buildings which cannot be immediately demolished and which have a gap of 1000 feet between them, obstruct the centre line of a proposed new road, which is to be perfectly straight. Describe in detail how the centre line could be set out in the gap between the buildings assuming that a theodolite is available. Illustrate your answer with a sketch and mention how you would check the accuracy of the work. (T.C.C.E.)

ARTICLE 11 : CONTOURS

Contours are imaginary lines through points of the same elevation on the earth's surface, the vertical distance between successive lines being known as the **contour interval**: 1, 2, 5, 10, 20, 50, or 100 ft., or 5, 10, 20, or 50 metres. There is no established ratio between the scale of the map and the interval, beyond conventional usage ; for example, 1 ft. or 2 ft. in city and certain constructional surveys ; 5 ft. in preliminary and parliamentary surveys, etc., 10 m. being often associated with the 1 : 250,000 map.

Contour lines are the chief convention of representing the topographical relief of the surface, being more precise than *hachures*, *shade lines*, or *altitude tints*, though these may be based upon contours.

The subject may be divided into (A) locating contours and (B) utilising contours, the following operations arising in the latter connection :

(*a*) Providing trial vertical and oblique sections, and giving information as to intervisibility of points, selection of routes and gradients, and the lateral slopes of the ground.

(*b*) Affording data for estimating earthwork and reservoir content, *directly* from a series of horizontal sections as determined by contours, or *indirectly*, by reciprocal contours, in calculating the earthwork to a final graded surface.

(*c*) Facilitating problems such as the intersection of the side slopes of cuttings and embankments with the existing ground surface, the supporting ground in a valley, and the position of the *grade contour* which gives neither cut nor fill on a specified gradient.

LOCATING CONTOURS

Contouring may be graded in accordance with the interval, which fixes the limit of error, and in the broad sense indicates methods and instruments accordingly.

Accurate	**Low Interval**: 1 ft. to 2 ft. (0·5 m.). City and precise constructional surveys.
	Standard Interval: 5 ft. (2 m.). Constructional, parliamentary, and precise topographical surveys.
Moderate	**Medium Interval**: 10 ft. to 20 ft. (5 m.). Preliminary, route, and extensive topographical surveys.
	Large Interval: 20 ft. to 50 ft. (10 m.). Pioneer, reconnaissance, and geological surveys.
	High Interval: 50 ft. to 100 ft. (200 ft.). (20 m. to 50 m.).

Approximate: Exploratory and geological surveys.

Crude: Form lines in military topography and reconnaissance.

Methods. Contours may be determined *directly* or *indirectly*, according as points are found on the actual contours, or are interpolated with respect to representative points of known but irregular elevation. Direct location is normally restricted to intervals between 2 ft. and 10 ft., commonly 5 ft., but where the ground exhibits definite surface features. Indirect location is used for small intervals on ground devoid of surface character, also for intervals upwards of 10 ft., direct location becoming too laborious in spite of its many advantages. Incidentally, what may be styled "pseudo-direct location" is used in connection with the stereo-photographic method, where the wandering mark, like a staffman, is made to follow a given contour on the relief, as viewed stereoscopically.

Contouring involves the third co-ordinate by which a point is fixed vertically in space by levelling, or **vertical control** in the general sense. **Horizontal control** denotes likewise the process of surveying the point, or fixing it by its two co-ordinates in the horizontal plane. When both con-

trols are conducted simultaneously with reference to a survey station, the parties work in **dual control**, but when they work independently, possibly in different sectors, the control is **detached.**

Direct vertical control is usually by spirit levelling. The collimation height is found, and improvised targets are attached to the staff at readings corresponding to two successive contours ; the staffman moves from place to place until he is halted by the surveyor when the cross-hair coincides with the target, a tolerance of $1\frac{1}{2}$ in. being permissible in the 5 ft. interval. In dual control the point is immediately fixed in horizontal control, but in detached control the contour points are suitably indicated, preferably by means of short pieces of lath, appropriately coloured for the various contours ; say, black for 50, green for 55, red for 60, and so on.

The indirect methods require that the elevations of a number of representative points, or **spot levels**, be known. When these are surveyed and plotted, the contours are interpolated, preferably as follows by means of strips of transparent squared paper, 2 in. or 4 cm. wide, the main lines being assigned even values. Thus the line corresponding to one elevation is passed through the relevant point, and the strip is turned about that point until the line corresponding to the other elevation passes through its plotted point, when the crossing of the main line is noted for the even contour value. This is more expeditious than the use of the sector or a radial diagram.

Ordinary methods of horizontal control consist in fixing the contour (or representative) point with respect to a traverse station by means of an angle (or bearing) and the horizontal distance to the point, which distance is taped, etc., or observed tacheometrically, vertical angles up to 6° being ignored when the vertical control is by spirit levelling. Wholly tacheometric methods involve the exact value of the vertical angle. In general, a tacheometer stationed beside a plane table is more economical than an elaborate complete alidade fitted with subtense lines.

Occasionally contour and representative points can be fixed by offsets to traverse lines run with the chain and the theodolite or compass, also in certain work plane table intersections may be used with advantage.

Special methods of horizontal control include the following : (i) unit squares ; (ii) cross-sections ; (iii) compass resection ; (iv) direction lines ; and (v) radial lines.

(i) **Unit squares.** Here the area is covered with a grid of squares, the sides of which range from 50 ft. to 200 ft., according to the interval primarily. Levels are taken at the corners of the squares, and the contours are interpolated between these, the process being simplified by the uniformity of the scheme. Incidentally, the grid appropriately figured with cuts and fills forms the basis of a systematic calculation of earthwork from the truncated prisms of which the squares are the plans. The

method is particularly convenient for 1 ft. or 2 ft. intervals, also for intervals from 5 ft. to 10 ft. on ground devoid of surface character.

(ii) **Cross-sections.** This American method applies particularly to contouring for preliminary railway surveys, though the principles can often be used with advantage. At the 100 ft. stations the reduced levels on the centre line are known, and hence the height of collimation when a hand level is strapped or otherwise attached to a 5 ft. Jacob staff. The staffman holding one end of a tape goes out at right angles to the centre line and is halted when the surveyor obtains a staff reading corresponding to a contour (Fig. 89). After this point (70) and possibly another have been found, the surveyor moves the Jacob to the outer (70) contour point, and

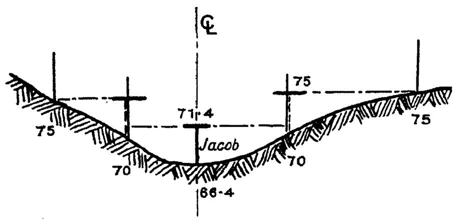


FIG. 89.

Left		C.L.	Right	
$\frac{75}{60}$	$\frac{70}{22}$	$\frac{66{\cdot}4}{0}$	$\frac{70}{24}$	$\frac{75}{65}$

afterwards merely looks for the foot of the staff, which at the ground gives the next (75) contour point. Distances right and left of the centre line are recorded as denominators, the numerators being the contour points, as recorded below Fig. 89.

(iii) **Compass resection** is used in exploratory work, the altitudes of salient points P being determined hypsometrically. The surveyor then observes three visible mapped points and reads the bearings of these with the compass, the differences giving the angles θ and ϕ in the three-point problem.

(iv) **Direction lines** are run along uniform slopes from the stations of a traverse, the bearings of the lines being taken with the compass and the slopes of the ground with the clinometer. The elevations of the stations of the traverse are ascertained with the Abney level and staff, and the contours are inserted with a slope scale, giving the relation $x = V \cot \alpha$, where V is the interval and x the distance between the contours.

(v) **Radial lines** afford a simple method of horizontal control wherever it is expedient to run radials from a central station to other stations or prominent objects in the survey ; or, alternatively, along directions at

observed angles, closing the horizon from 0° to 360°. Contour, or representative, points are taken on the radials, and the corresponding distances from the central station are chained or observed tacheometrically.

Contour location is open to a wide choice of methods, and so tests the surveyor's ability in the economic choice of equipment and procedure. In order to review the subject concisely the table on p. 180 is given in preference to descriptive text.

UTILISING CONTOURS

Slope scales. A slope scale is a ready and improvised means of determining the slope of the ground from contoured maps ; or, in other words, a means of finding the angle of slope α as given by the relation $H = V \cot \alpha$, where V is the contour interval and H the horizontal equivalent—or distance—between successive contour lines on any section. The chief use of the scale is that of ascertaining transverse slopes when computing the areas of cross-sections in preliminary estimates, though modifications of the scale may be used in various other connections.

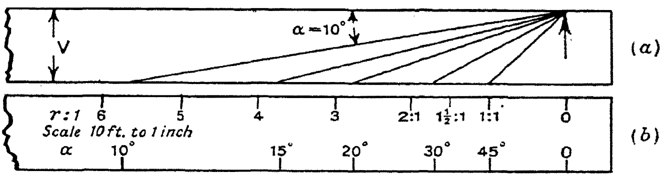


FIG. 90.

For small scales, where V is considerable, or great accuracy is not required, the slope scale may be constructed with the aid of a protractor, the angles being set off from the upper line of Fig. 90 (*a*), the width of the strip being the interval V on the common scale of horizontal and vertical distances. The points thus determined on the lower edge are projected to one edge of another strip (of any convenient width) and figured with the corresponding vertical angles α, as in Fig. 90 (*b*). It is desirable also to show on the other edge of the strip the corresponding horizontal equivalents, expressing the slopes conveniently as r horizontally in 1 vertically.

For medium scales, such as are used in preliminary maps, or when V is small comparatively, the scale is best constructed with the aid of a table of cotangents, angles from 5° to 30° being included. Thus, if s is the scale of the map, s ft. to 1 in., and the interval V is in feet, the distances from the index of the scale will be given by

$$H = \frac{V \cot \alpha}{s} = \frac{Vr}{s} \text{ in.}$$

METHODS OF CONTOUR LOCATION

Interval / Mode	Horizontal control	Vertical control
1 ft. to 2 ft. *Indirect.*	Unit squares ; 50–100 ft. side.	Spirit levelling.
5 ft. (2 ft. to 10 ft.) *Direct.*	(i) Theodolite angles (bearings) from traverse stations to contour points with either chained or stadia distances.	Spirit levelling with improvised targets on staff.
Direct.	(ii) Plane table radiation with chained or stadia distances, or intersections occasionally.	,,
Direct.	(iii) Chain and theodolite (compass) traverses with offsets to contour points.	,,
Direct.	(iv) Cross-sections to centre line of trial route.	Hand level on 5 ft. Jacob staff.
Indirect.	Unit squares ; 50–100 ft. side.	Spirit levelling.
10 ft. (5 ft. to 20 ft.)	(i) Unit squares; 100–200 ft. side.	Spirit or tacheometrical levelling.
Indirect.	(ii) Ground photography.	Stereocomparator, autograph, or stereoplanigraph.
Indirect.	(iii) Angles (bearings) with tacheometrical distances.	Tacheometrical levelling. (Spirit levelling up to 10 ft.)
Indirect.	(iv) Ground photography.	Elevations from y co-ordinates.
Indirect.	(v) Plane table radiation (or intersections).	As in (iii).
20 ft. to 50 ft. *Indirect.*	(i) Ground photography.	Graphical photo-elevations.
Indirect.	(ii) Direction lines by compass (or plane table).	Clinometer angles on uniform slopes.
Indirect.	(iii) Aerial photography.	Contours by stereoplanigraph (or aero-projector) (10 ft. with extensive ground control.)
50 ft. to 100 ft.	Compass resection or plane table.	Hypsometry.
Indirect.	(i) Plane table intersections.	Slopes by clinometer
Indirect.	(iii) Aerial photography.	Simple stereometer methods.

Plotting slope limits. Whenever it is desired to show the complete plans of cuttings or embankments, the outlines of these may be determined on large-scale plans by the intersections of the regular contours of the side slopes with the irregular topographic contours of the map. Consider the plan of a road, as shown in Fig. 91, where the formation is shown with formation levels figured at cross-sections, *C.S.* 1, *C.S.* 2, etc. Slope contours at 5 ft. intervals are inserted by considering the rise or fall from formation level at any given section, and calculating the corresponding horizontal distance out from the edge of the formation. Thus at *C.S.* 1 the edge of the formation gives the 20′ slope contour, and if the batter is $1\frac{1}{2}$ to 1, the 15′ or 25′ contours will be $5(1\frac{1}{2}) = 7{\cdot}5'$ horizontally outwards on the scale of the map, while the adjacent contours, the 10′ or 30′, will be a like distance, and so on, outwards on the cross-section. If the road

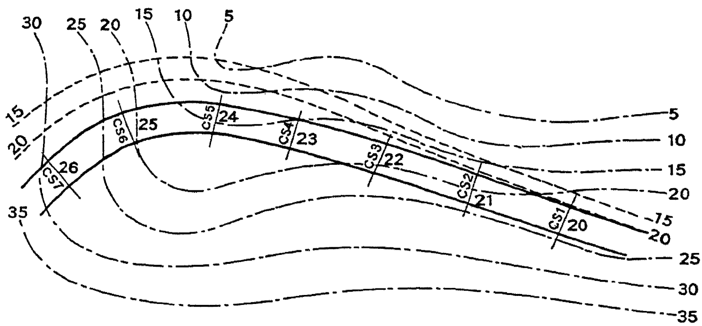


FIG. 91.

is horizontal, the slope contours will be parallel lines parallel to the centre line on straight runs, and concentric circles on circular curves ; and if the road is on a gradient the contours will remain parallel on the straight, but not parallel to the centre line, while on circular curves they will deviate from concentric circles, though they may be drawn as such on easy gradients. Thus it is necessary only to mark slope contour points at the tangent points and at possibly one or two intermediate points, and to join these by fair curves, if not circles.

The outline on which the side slopes run into the existing surface of the ground is found by the intersection of the slope contours with the surface contour, the 15′ with the 15′, the 10′ with the 10′, and so on. This offers no difficulty where several such intersections occur, as at the left of Fig. 91, but where the contours take the trend of the road, as at the right of Fig. 91, it is necessary to interpolate intermediate contours both on surface and slopes, or to outline the cross-section and on this determine the slope limits.

The last process may be obviated by the use of close divided scales in the ratio of 1, $1\frac{1}{2}$, or 2, to that of the scale of the map, or the last two may be constructed on a strip of paper. The reading on the scale corresponding to formation level is placed at the edge of formation along a cross-section, and a point is found at an estimated ground level identical with the scale reading at this point. Thus if formation level is 35·0, this value on the scale is set at the edge of formation, and on inspection it may appear that between the 20′ and 15′ contours, 17′ on the scale corresponds with an estimated ground level of 17′ also.

When the contours are a considerable distance apart it is often more expeditious to find the lateral slope on the cross-section between contours, and to find the slope limits by means of the rules for half widths :

$$W_s = \frac{sD}{1 \pm s/r}.$$

(*a*) **Supporting ground.** In certain cases of railway construction, especially on hill-side ground, it is possible to throw the location to one side or other of the preliminary traverse, taking advantage of the so-called "supporting ground", thus reducing the earthwork to a practical minimum with approximate balance of cut and fill, having due regard to appropriate working gradients. The final location, however, will normally be confined to definite limits within the strip mapped, and the possibilities of the ground should be investigated prior to finding the grade contour, which would give the ideal route were the minimising of earthwork quantities the only economic consideration.

Let the thick irregular line *PQ* in Fig. 92 represent the surface of the longitudinal section (or profile) and the dotted lines the profiles of the upper and lower edges of the preliminary zone, *AB* being the datum of the section.

One-gradient section. Let *PQ* be the initial and terminal points as joined by a single grade line. Here it is evident that the fills are greatly in excess of the cuts ; but between *q* and *r*, where the fill is greatest, the ground within the strip rises above the grade line, and by throwing the location towards the uphill side, it can be made to rest upon ground which

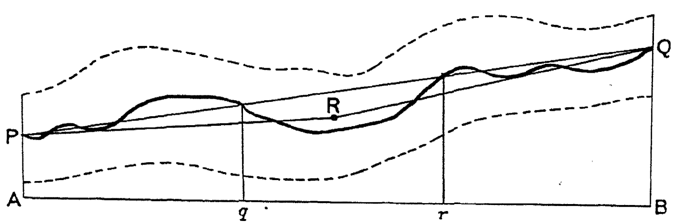


FIG. 92.

follows the straight grade line closely without passing outside the limits of the mapped strip. Also it will be seen that the location will deviate appreciably from the traverse ; and if the conditions are then circumscribed, a two-segment grade line, as *PR Q*, may be desirable or necessary.

Two-gradient section. Now the conditions may be similar to those shown in Fig. 93, where a single gradient would involve a very great excess of fill, and the limits of supporting ground would not permit the location to be thrown uphill in order to minimise the earthwork.

A two-segment gradient, such as *PR Q*, may appear to conform with the best location, the latter not deviating far from the preliminary traverse, but, on inspection, under the exaggerated conditions indicated, it will be seen that so steep a gradient as *RQ* would not be admissible, and, in consequence, the full use of supporting ground would be advisable, reducing the gradient, as in *PSQ*.

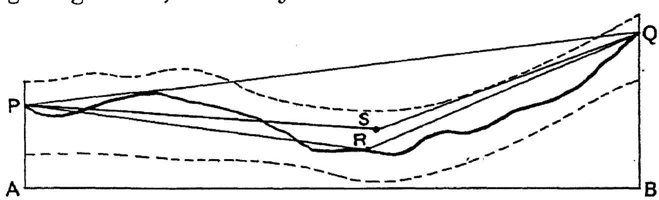


FIG. 93.

Three-gradient section. Further, the configuration of the ground may be such that three different gradients should be used so as to conform with constructional requirements within the portion *PQ* of the location considered. Tentatively, though by no means generally, the best location will be obtained by using as few breaks in gradient as possible, making the entire grade line as nearly as may be a straight line, with the gradients reasonably below the ruling limit and the grade line between the upper and lower profiles, which incidentally may be definitely prescribed limits.

(*b*) **Grade contour.** A grade contour is a line of given slope (or slopes) which lies wholly on the surface of the ground between any two given points. Grade contours are used in planning the location of a railway or highway. Since their trend involves little or no cut or fill, they provide a basis of the location, subject, however, to such deviations as constructional or economic considerations demand.

Let it be required to determine the grade contour for a specified gradient g between two points P and Q, the interval of the contours being V. Here the process merely consists in opening a pair of dividers to a distance apart equal to the distance to scale along the gradient necessary to climb from one contour to the next, striking arcs with this radius r, and

using the intersections with the contours successively as the centres of arcs. Thus if the scale is s ft. to 1″, and g is the denominator of the gradient, the dividers will be opened to a radius $r = Vg/s$ in. for successive contours, while if the starting point is not on a contour, but is z ft. below, the radius for the first arc will be $r' = zg/s$.

Consider the procedure with regard to Fig. 94, where the gradient $1/g$ is 1 : 75 and the contour interval 5 ft.

(1) Open the dividers to 375 ft. by scale; that is, 75 times the interval of 5 ft., and with P as centre, strike an arc cutting the 35′ contour q.

(2) Using the same radius and centre q, determine r, and so on until Q, or a point nearby, is reached.

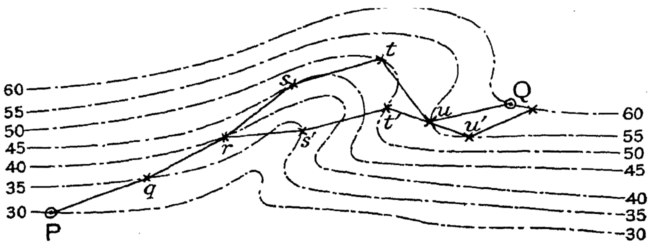


FIG. 94.

Now in railway location the bends t, u, would not be permissible, since the grade contour should not be carried round ridges or up ravines, thus necessitating unduly sharp curves. Hence amend the procedure as follows:

(3) Carry the grade contour from r in two steps, rs', $s't'$ to t', which is on the second contour above r.

Usually two or more gradients will occur in the distance PQ, and the dividers must be set at each break in the gradient, a new radius being found by the rule $r = Vg/s$. Obviously the grade contour will be curved since the contours are curved, though in practice it is regarded as a succession of broken lines, which are straight between contours.

The laying down of the centre line of a proposed railway—the paper location—consists in approximating to the grade contour with straights and curves, the process of finding the best location being a matter of trial and error. If the tangents are relatively long, these are laid down first and the curves are fitted to them; but if the tangents are very short, the curves are laid down first, and the tangents are fitted to these. The procedure varies with the terrain, practical considerations as to minimum length of tangent and selecting curves, and predetermined considerations, which are strictly a part of railway engineering.

QUESTIONS ON ARTICLE 11

1*. You are required to contour to a 1 ft. interval an area of about 30 acres of undulating ground which exhibits no definite surface features, yet embodies differences of elevation up to 18 ft.

Detail concisely how you would proceed with the following equipment and personnel at your disposal:

Theodolite (without subtense lines), dumpy level, levelling staff, range poles, whites, chain and arrows.

Trained assistant, staffman, and two chainmen. (U.L.)

2*. Describe concisely, giving sketches and specimen notes, how you would contour to a 5-ft. interval a theodolite survey of about 40 acres of ground which exhibits well-defined hill and valley features, the following equipment and personnel being at your disposal:

Tacheometer, dumpy level, 2 levelling staves, whites and laths.

One trained assistant and two handymen. (U.L.)

3*. You are required to insert the 5-ft. contours of a survey which is plotted on continuous sheets suitable for the board of a plane table.

The area under survey exhibits definite hill and valley features, and it has been suggested that the contours should be plotted in the field.

Describe concisely how you would proceed with the following equipment and personnel at your disposal:

Plane table with telescopic alidade, theodolite with stadia lines, dumpy level, two levelling staves, whites and laths.

Two trained assistants and two staffmen. (U.L.)

4*. You are required to contour to a 5-ft. interval a mountain valley which is to be used as an impounding reservoir, the valley being reasonably straight for the area under consideration.

Describe concisely how you would proceed with the following personnel and equipment at your disposal:

Two trained assistants, two chainmen, and two staffmen.

Theodolite with stadia lines, 2 dumpy levels, 2 levelling staves, range poles and 100 ft. chains and tapes. (U.L.)

5*. You are in charge of a topography party on the preliminary survey for a colonial railway, and you are required to locate the 5-ft. contours on each side of the traverse run by the theodolite party.

Describe how you would proceed with the following equipment and personnel at your disposal, giving sketches and specimen field notes:

Hand level, levelling staff, tapes, and 5 ft. Jacob staff.

Staffman and two chainmen.

6*. You are required to contour to a 10-ft. interval the broken course of a river valley, the country being open but difficult.

Detail concisely, giving sketches and specimen notes, how you would proceed with the following equipment and personnel at your disposal.

Zeiss level, 2 Jeffcott self-reducing tacheometers, and 2 levelling staves:

One trained assistant and two staffmen. (U.L.)

7*. State clearly the points of similarity between plane tabling by "intersections" and ordinary photographic surveying in particular regard to the plotting of points and the determination of elevations, the Abney level being used as an auxiliary instrument with the plane table.

Summarise the factors that would influence your choice between the above methods in a rapid survey of open mountainous country, accuracy being desirable but subordinate to speed. (U.L.)

8*. You are required to run levels and contour a valley feature where a lane is to be developed into a by-pass road 2 miles in length. Describe clearly how you would proceed with the following personnel and equipment at your disposal :

Assistant and four council employees.

Theodolite, 2 engineers' levels, 1 hand level and 5-ft. Jacob staff, range poles, chains, arrows and tapes. (U.L.)

9*. You are required to run a rapid survey of about four miles through a wide, open mountain valley, the completion of the map being urgent. Summarise the factors which would influence your choice between plane tabling and ordinary photographic surveying, and outline the procedure of the method adopted, including the determination of elevations.

Equipment. Plane table, Abney level, geographers' theodolite, surveying camera, aneroid, passometer, and tape.

Personnel. Surveyor and two trained assistants. (U.L.)

10. Give with necessary explanations some of the ways in which cumulative errors may arise in levelling.

Define a contour line, a level surface, new ordnance datum.

Explain how contours may be obtained by : (*a*) Cross-sections with a level ; (*b*) gridding with a level ; (*c*) "running a contour line" ; (*d*) a hand level. Show a sample booking for this case. (U.B.)

11. Describe the various methods of contouring an area, stating which method you would adopt in a preliminary survey for a road or a railway. (U.D.)

12. Describe fully how contours on both sides of a previously surveyed and levelled line may be obtained, using a hand level. Give part of a sample page of the field book. The contours are to be at 5 feet intervals. Make a complete list of the necessary apparatus. (U.D.)

13. Describe the various surveys that have to be made previous to the construction of a railway or road through unmapped country. Give full details of the work done during each survey. (U.D.)

14. A piece of ground to be developed for a building site is approximately square and has an area of about 30 acres. How would you proceed to make a contoured plan of the area? The contour interval is to be 5 ft. (U.D.)

15. A contoured plan of a long narrow strip of country on both sides of a proposed railway is to be prepared. Describe how you would use a hand level or a clinometer in locating the contours and show a sample page of the field book. (I.C.E.)

16. A road is to be located to connect two points on opposite sides of a range of hills with steep slopes, and the maximum allowable gradient has been fixed. Describe concisely the steps you would take to discover the best route. (I.C.E.)

17. Describe the method of carrying out the preliminary survey for a railway in a country of which there are no reliable maps, indicating in detail the work of the various groups of the survey party. (I.C.E.)

18. Let it be supposed that a piece of ground of about the size and shape of the College grounds, but having a maximum difference of level of 50 feet, is to be laid out as a public park. If you were required to make a contoured plan with a 5 feet interval, describe in detail how you would perform the fieldwork. (U.D.)

19. Describe briefly one method of contouring to 5-foot V.I. on a plane table on the scale of 100 feet to 1 inch in undulating country. (Slope of ground does not exceed 1 in 20). (T.C.C.E.)

20. You are placed in charge of the canalisation of an area in the plains of India. Topographical maps on the scale of 1 inch to a mile, with 50 ft. contours, are available. Give details of what further survey you will require, and on what points you will collect information. (T.C.C.E.)

ARTICLE 12 : MISCELLANEOUS PROBLEMS

A number of elementary problems in engineering surveying will be revised in this, the concluding article, the following summaries serving as an introduction to the examples.

LEVELLING

The unqualified use of the term "levelling" suggests spirit levelling, though strictly the subject may be divided into three categories : (i) **gravitational levelling**, most commonly with the spirit level, but occasionally with the reflecting level and the water level ; (ii) **trigonometrical and tacheometrical levelling**, based upon angular relations with a line fixed fundamentally by spirit levelling ; and (iii) **hypsometrical levelling**, based upon variations in atmospheric pressure as observed through the medium of the barometer and boiling point thermometer.

(1) **Levelling difficulties.** The following are among the common difficulties encountered in spirit levelling :

(*a*) *Ascending or descending a hill*, which involves sights near the foot and the top of the staff, with the foresight about one-third the length of

the backsight, and *vice versa*, thus introducing numerous settings-up of the level and errors accumulating from minor defects in the line of collimation. Uncertainty of reading near the top of the staff owing to faulty verticality is obviated by causing the staffman to wave the staff (about its foot as centre) to and fro in the vertical plane of the line of sight and recording the lowest reading observed. As in tacheometry, a staff bubble is useful in this connection, provided its adjustment is tested from time to time. The balancing, or equalising, of the lengths of the backsights and foresights is often impossible in the circumstances, and very short sights should be avoided by setting up the level to one side of the line, taking thus zig-zag sights.

(*b*) *Staff station* (*or B.M.*) above the line of sight presents no difficulty when it is possible to invert the staff on the point, the fact being noted and the staff readings prefixed with the minus sign, and treated as negative values. Often a bench mark is so little above the line of sight that it is impossible to invert the staff; and in this case a reading is taken on the ground below the *B.M.*, the negative reading being the staff reading subtracted from the measured height of the *B.M.*

(*c*) *Lakes and ponds*, too wide to be sighted across, can be regarded as single change (or turning) points by driving pegs flush with the *still* water surface. Similar cases with rivers are best dealt with by reciprocal levelling when the sights exceed ten chains.

(*d*) *Board fences* offer no difficulty if a nail or spike is driven through to act as a change point, serving thus both sides.

(*e*) *Brick walls* are best treated by driving pegs in the line against the wall on either side (if the ground is soft), measuring up from these to the top of the wall, and introducing the top as a virtual change point in conjunction with fore- and back-sight readings taken on the pegs.

(2) **Curvature and refraction.** Two problems associated with long sights are curvature and refraction and reciprocal levelling, the last being a process seldom assessed at its real value. In the present connection considerations of curvature and refraction are largely academic, since in ordinary work the errors of reading the staff are greater than the resultant effect, $(c-r)$. The effect of the earth's curvature c is involved when the length of the horizontal line of sight is so considerable that it may no longer be regarded as identical with the level line of the earth's curvature as determined by mean sea-level, the effect being to make points appear lower than they really are, and thus increasing the staff reading. Atmospheric refraction is the bending downwards of rays of light in passing from a rarer to a denser medium, thus reducing the reading of the staff and in part eliminating the effect of curvature. The effect, r, is more conventional than certain in the case of horizontal sights, but is usually taken at $\frac{1}{7}c$, following from a coefficient m of 0·07 applicable to the angle subtended by the sight at the earth's centre (see p. 192).

Reciprocal levelling is resorted to when very long sights are necessarily involved. Sights are taken first from A on one side of a valley, etc., to B, and then from B, on the other side, to A, the mean difference of reduced level being the true difference, with the effects of curvature, refraction, and collimation error eliminated. A target staff should be used in the operation, since it is seldom possible to take a direct staff reading ; and the mean of three or four readings should be taken in either direction. Sometimes the level is set up as near as may be to the peg at the staff station, A or B, measurement up to the eyepiece serving as one reading ; and sometimes the level is stationed at equal distances s from the pegs, s being greater than the minimum sight length. In the former case the combined errors of curvature and refraction and maladjustment can be found from the difference of the reciprocal differences of elevation, and so if the sight lengths are short, the pegs A and B will provide the basis of the two-peg test in the absence of a chain.

When the differences in elevation exceed the range of a levelling staff, about 10 ft., reciprocal sights may be taken with the theodolite or tacheometer.

(3) **Boning-in.** When two pegs A and B are driven with their tops *at*, *above*, or *below* any required gradient, the surveyor can station himself at A and, sighting across the top of that peg to the top of B, direct the driving of intermediate pegs to the gradient thus established. Boning rods are used in various connections. These usually consist of three T-shaped rods, 3 ft. or 4 ft., in height, two being fixed and the other moved from point to point with its top in the gradient of the tops of the cross-pieces of the other rods.

In the case of trenches, sight rails are erected, consisting of the horizontal sighting rail and its two supporting posts, which are driven into the ground, one on each side of the trench, or otherwise are fixed in drain pipes stabilised with earth packing. Pegs or nails of known reduced level are inserted near or at the posts, and the rail at the lower level is fixed at a convenient height, which is above the invert level of the drain at a distance equal to the length h of the boning rod. The height of the next (higher) rail is found by adding the length h to the invert level at that point, and thence the height of the rail above the peg thereat is readily known.

(4) **Slope limits.** Slope stakes are driven to indicate the points in which the side slopes of a cutting or an embankment will meet the existing surface of the ground, the operation being important in the control of earthwork. A similar process is used in connection with benchings and graded formations. The British practice in line work is by trial and error, since two unknowns are involved, the side width W_s from the centre line and the appropriate staff reading R at the stake. Occasionally the process is facilitated by scaling from a cross-section. In American practice the work is incidental to the method of cross-sectioning, in which

areas are calculated in the field, and, occasionally, the driving of slope stakes is made more or less "automatic" by the use of special rods and slope tapes (see p. 196).

(5) **Tacheometry.** Tacheometric methods are often used with advantage in preliminary work, but the method demands careful manipulation and control of errors, or the results can be very disconcerting so far as levelling is concerned. Tacheometrical levelling can be most economical in difficult country where numerous settings-up of a level would be tedious and would impair the general accuracy of the work. The process is the same in all systems as regards the reduction of elevations, a vertical component (or distance) being involved from the vertical angle.

In the *collimation system* the reductions can be made expeditiously if V is prefixed with the *plus* or *minus* sign according as α is an angle of elevation or of depression.

Collimation heights (H) are less than those of spirit levelling by the algebraical value of the vertical component V.

Elevations (E) exceed the reduced levels of spirit levelling by the algebraical value of V.

Backsights : $H = E + \text{B.S.} - V$; Foresights : $E = H - \text{F.S.} + V$,
(Intermediates)

where E is the elevation of the staff station.

(6) **Compass surveying.** The position of the compass in modern surveying has been discussed in Sect. I (p. 87), where the case of working in the presence of magnetic interference was mentioned. In this contingency it is necessary to observe both the back and forward bearings at a station, since, however affected, the needle will give a fiducial line from which the two parts of the affected angle can be found, the sum giving the true magnitude.

(7) **Hypsometry.** Barometrical levelling is used in pioneer and exploratory surveys for determining approximate altitudes (p. 84). Unless a first-rate instrument is used with extreme care and due regard of corrections, the results can be extravagant. The use of batteries of aneroids has led to a marked improvement in the accuracy of this method of levelling.

(8) **Field engineering** is the converse process to surveying, and is commonly known as "setting-out works"; among which may be cited (*a*) railways and highways, including circular, vertical, and transition curves, with bridge abutments and tunnel alignments, etc.; (*b*) public works, including water supply, main drainage, irrigation, etc., with the numerous problems in gradients, boning-in, planning sites, etc. Common to most are earthwork and other estimates.

In general, a preliminary or parliamentary plan and estimate are made, the common scales being respectively those of the "six inch" Ordnance

sheet and 200 ft. to one inch. After the location or lay-out has been planned, the data are taken from the map and set out on the ground, any variations or additional information being embodied in the final or location survey, which is often on the twenty-five inch sheet in Great Britain, and 200 ft. (100 ft.) to one inch in North America.

Variations are made, particularly in regard to constructional plans. Also the scales for vertical sections vary to some extent. One inch to 100 ft. is used in parliamentary surveys on the 6-inch scale, while 1 inch to 20 ft. or 40 ft. may be used with a horizontal scale of 200 ft. to 1 inch.

Legally, the Gunter chain is the unit in the United Kingdom, though the 100 ft. unit is used in certain connections. This latter is used exclusively in engineering surveys in North America, where in line work the even number of 100 ft. units precedes a "plus", which gives the number of feet and fractions: thus, 121 + 36·7 for 12,136·7 ft.

OBSERVING ANGLES

It is desirable to recapitulate the methods by which angles are observed in engineering surveys, namely, (*a*) direct angles; (*b*) deflection angles; (*c*) bearings.

Direct angles are favoured in the United Kingdom, not only for triangulation but also for traverse surveys. These are clockwise 0° to 360°, however the alidade is turned, on account of the division, and so occasionally, as in "curves to the left", the vernier has to be read backwards or the angle as read subtracted from 360°. Direct angles have the advantage that each angle can be measured separately and may be repeated a number of times if desired with reversed faces of the theodolite, thus eliminating errors in horizontal angles due to maladjustments.

Back angles are a modification of direct angles in which the angle is measured from the back station of a line, and whether the circle is turned clockwise ("swing right") or anticlockwise ("swing left"), the clockwise angle from 0° on the back station will appear on the circle. Back angles are used not only in line work and town surveys but also in closed traverses, which if traversed in the counterclockwise direction will give the interior angles of polygons. Such angles need reduction to bearings in order that latitudes and departures may be calculated.

(*b*) **Deflection angles** are used in railway and route surveys in America, where the "half-circle" division (0°–180°–0°) is added to facilitate their use. A zero sight (*F.R.*) from O is taken on the rear station P, the telescope is transitted (*F.L.*) and a sight is taken on the forward station Q, the vernier showing the deflection right or left of the direction PO produced, as 36° 15′ *L.* or 27° 32′ *R.*

Transitting (or "plunging") the telescope is not favoured in the United Kingdom on account of the fact that the effect of errors of adjust-

ment are amplified unless observations with both faces are used; this is not possible in the following process, which also precludes the mean of both verniers to eliminate eccentricity of the circles. On the other hand, the work is simplified and expedited, provided the surveyor keeps his instrument in perfect adjustment. Also if transitting is avoided by the use of plate reversals (that is, re-setting the vernier through 180°) the additional work is as tedious as that of reducing back angles to bearings.

(*c*) Bearings may be *true*, *magnetic*, or *arbitrary*, according as they are measured from the true, the magnetic, or any assumed meridian. When they are recorded from 0° to 360°, as in military surveying, they are alternatively known as azimuths or "whole circle bearings" (W.C.B.), and when they are observed or reduced to the quadrant system (0°–90°–0°–90°–0°) they are styled bearings, or "reduced bearings" (R.B.), being recorded as N., E., S., W., etc. Azimuths are necessarily observed with British instruments, but in North America the quadrant division is often added to the whole circle division, thus facilitating the immediate use of latitudes and departures.

The fixed needle method of observation is briefly as follows: First sight 0° (*F.L.*, *L.M.*) from *A* along meridian, sight (*F.L.*, *U.M.*) next forward station *B*; proceed to *B*, and sight back on *A* (*F.R.*, *L.M.*) with circle clamped to reading obtained on *B*; and proceed, sighting forward stations (*F.L.*, *U.M.*), and backward stations (*F.R.*, *L.M.*), with the vernier clamped at the forward reading on the station occupied. On again occupying Station *A*, the error of closure of the polygon appears at the vernier used exclusively (the *A* vernier normally); though *A* need not be occupied if initially the preceding station is sighted (*F.R.*, *U.M.*) after sighting along the meridian, but the supplement of this bearing must be noted.

In general, the method with magnetic azimuths or bearings should be avoided when filling in the details of triangulation surveys with different theodolites, as discrepancies in the needle at the first bearing will lead to divergencies which are tedious to correct by a reference sight on a common station. Also the method generally precludes the use of the mean of both verniers in eliminating eccentricity.

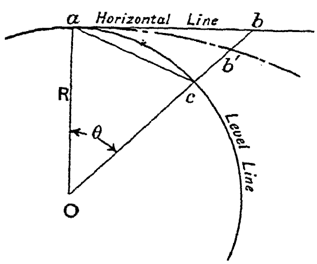


FIG. 95.

*Example**. Derive an expression for the correction for curvature and refraction in spirit levelling, assuming a coefficient of refraction of 0·07.

Let *ab* represent the horizontal line of sight of the instrument. This is tangential at *a* and gives the apparent true level, while the true level is *ac*,

which is a line concentric with the earth's mean curvature, O being the earth's centre and R its mean radius, say, 7913 statute miles (Fig. 95).

$(ab)^2 = bc(bc + 2cO)$, and since bc is small in comparison with $2cO$,

$(ab)^2 = bc \times 2cO$ without sensible error.

Whence the curvature correction :

$$bc \approx \frac{(ab)^2}{2cO} = \frac{1 \times 5280 \times 12}{7913} \approx 8'' \text{ per ml.}^2 ;$$

or $$c = 0{\cdot}66M^2 \text{ ft. with } M \text{ in miles.}$$

The refraction effect is $m\theta$, or $2m(bac)$ with the angle bac in circular measure, which for small angles is represented by the distance bc. Thus

$$r = bb' = 2(0{\cdot}07)\ bc.$$

Hence $$c - r = 0{\cdot}57M^2 \text{ ft.}/(\text{mls.})^2.$$

*Example**. In connecting a circuit of levels across a wide river, a distance exactly 1320 ft. between pegs A and B, the level was set up at A and the height of its eyepiece observed, being 4·62 ft. above the peg, the reduced level of which was 50·38. Sights were then taken across to a target staff held on the peg B, and the mean of four readings was 6·038. The level was then set up likewise at B with the eyepiece 4·49 ft. above this peg, and the mean of four readings on the staff held on A was 2·964.

Determine (a) the true reduced level of the peg B, (b) the error in the collimation adjustment of the level, and (c) the errors due to curvature and refraction, assuming reasonable values for the relevant corrections.

(U.L.)

From A : true diff. of elev. $= A \sim B = (b_1 + E) - a_1 = (b_1 - a_1) + E$. ...(1)

From B : true diff. of elev. $= A \sim B = b_2 - (a_2 + E) = (b_2 - a_2) - E$. ...(2)

Adding (1) and (2) eliminates the total error E, which embodies the errors of (b) and (c) (Fig. 96) :

(a) True diff. of elev. $(A - B) = \frac{1}{2}((b_1 - a_1) + (b_2 - a_2))$

$$= \tfrac{1}{2}((6{\cdot}038 - 4{\cdot}62) + (4{\cdot}49 - 2{\cdot}964))$$

$$= 1{\cdot}472 ;$$

and reduced level of $B = 50{\cdot}38 - 1{\cdot}472 = 48{\cdot}908$.

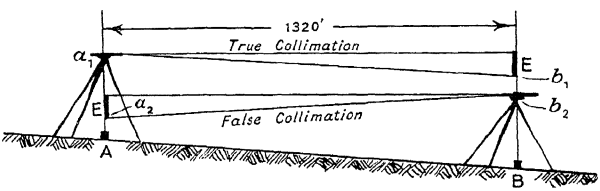


FIG. 96.

(*b*) Now E, as suggested by the A to B readings, tends to reduce the staff readings, and the collimation error will be greater than E by the effect of curvature and refraction, which must be determined first.

(*c*) Curvature correction c at $8''$ per (mile)^2 to be made on $\frac{1}{4}$ mile, $c = 8''(\frac{1}{4})^2 = \frac{1}{2}'' = 0{\cdot}042$ ft., tending to increase staff reading (+). Refraction correction r, usually $\frac{1}{7}$ of $c = 0{\cdot}006$ ft., tending to decrease staff reading (−). Whence $c - r = 0{\cdot}036$ ft. Subtracting (2) from (1),

$$1{\cdot}418 - 1{\cdot}526 = -0{\cdot}108 = -2E\ ;\ E = 0{\cdot}054 \text{ ft.}$$

Collimation error $= 0{\cdot}054 + 0{\cdot}036 = 0{\cdot}09$ ft. in 1320 ft.

Example†. Derive expressions connecting the half-widths and corresponding staff readings in the process of "driving slope stakes", or setting out the limits where the sides of cuttings or embankments will meet the existing surface of the ground.

Discuss any one artifice that has been employed to facilitate this operation. (U.L.)

The solution of this problem must be by trial, since it involves two unknowns W_s and R, as will be evident from Figs. 97 and 98, which represent respectively a cutting and an embankment in cross-section, w being the half-formation width, d the centre-line cut or fill, r the staff reading on the centre line, s the side slope ratio (s horizontally to 1 vertically) and R the staff reading at the slope stake S concomitant with the side width W_s.

Now in Fig. 97 (left) $W_s = CU = CT - UT = w + sd - s(R - r)$,

(right) $W_s' = CU' = CT' + U'T' = w + sd + s(r - R)$.

Hence for cuttings generally, $W_s + s(R - r) = w + sd = W$, ...(1)

where W is the level half-width.

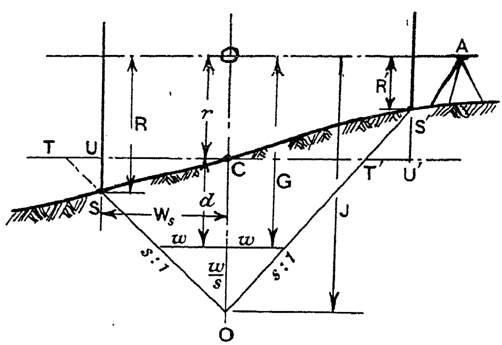


FIG. 97.

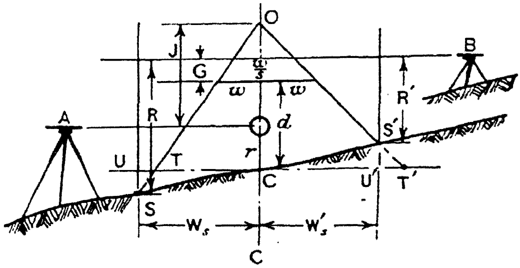


Fig. 98.

Also in Fig. 98 (left) $W_s = CU = CT + UT = w + sd + s(R - r)$,

(right) $W_s' = CU' = CT' - U'T' = w + sd - s(r - R)$.

Hence for **embankments** generally, $W_s - s(R - r) = w + sd = W$. ......(2)

Ordinarily in British practice the process consists in calculating W, the half-width on ground-level across, marking this, and estimating the position of S as judged by the rise and fall in the ground ; then reading a staff held at the assumed position of S, and observing if W_s as measured hereto accords with R in the relation $W_s \pm s(R - r) = W$.

The relation expressed in (1) and (2) may be written :

for *cuts* and *fills* respectively, where J is the difference between the height of collimation and the elevation above datum of the apex of the formation triangle resulting from producing the side slopes to meet the centre line above or below formation. (Incidentally the distance G from formation to collimation is the "grade rod" used in certain American methods of cross-sectioning). Accordingly sJ is $w + s(d + r)$ for cuttings and $w + s(d - r)$ and $w - s(d - r)$ for high and low banks, as shown respectively on the left and right of Fig. 98.

Various ways of simplifying the process may be employed : (1) Diagrams or improvised scales, utilising Eq. (3) ; (2) special slope tapes with the zero graduation at the half-width of the road bed and the remaining graduations to suit the slope ratio, $s : 1$; and (3) special levelling rods, fitted with endless, sliding graduated bands.

Example†. Give expressions connecting the half-widths and corresponding staff readings in the process of driving "slope stakes", or setting out the limits where the side slopes of cuttings and embankments will meet the existing surface of the ground.

Describe in detail some artifice which renders the above operation "automatic". (U.L.)

In this American method, the following are employed :

(*a*) Levelling rod fitted with a band, 20 ft. in length, which passes over friction rollers at the ends of a 10 ft. rod, the figures running up the front and down the back continuously to 20 ft.

(*b*) Slope tape with *O* at half-formation width w and divisions giving the slope ratios $s : 1$.

The formulae of pp. 194/195 are expressed as follows :

Cuts. $$\frac{1}{s}(W_s - w) = k = -R + d + r.$$

Fills. $$\frac{1}{s}(W_s - w) = k = R + d - r,$$

the term k being obtained with the slope tape when the zero is w ft. out on the appropriate slope scale ; $s = 1,\ 1\frac{1}{2}$, or 2.

Cuttings. Set d at the front bottom zero and take the centre-line reading r on the front of the rod. Bring r to the front bottom zero, and read R on the back.

Although the limit of d is normally 10 ft. a revolution of the tape will count as 20 ft. ; that is, 20 ft. is added mentally to both k and R. Thus if d is 35 ft. and r is 2 ft. in a cutting, $k = 37 - R$; that is,

$$\frac{20}{s} + k' = 37 - (R + 20) = 17 - R.$$

Hence 17 is brought to the bottom zero, and at the back 7 is at the top and 17 at the bottom. Thus if the correct reading for R is 9, 8 would be read directly in front while the reading thus obtained will be 9 on the back.

Embankments. Set d at the front bottom zero and take the centre-line reading r on the back. Bring r to zero at the bottom and read R on the front.

Example†. Describe the "double tape" method of driving slope
*

Here an ordinary levelling staff is used in conjunction with two ordinary linen tapes (*B* and *C*) and a 25-ft. length of slope tape fitted with end clips (*A*) (Fig. 99).

In the following procedure the upper and lower notation refer to cuttings and banks respectively.

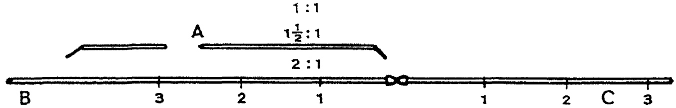


FIG. 99.

* "A Simplified Method of Driving Slope Stakes", A. L. Higgins, *Railway Engineer*, 1923.

(1) Take a reading r on the centre line (C.L.)

$$\left.\begin{array}{l}\text{Add } r \text{ to } d \\ \text{Subtract } r \text{ from } d\end{array}\right\} \text{ multiply } \left\{\begin{array}{l}(d+r) \\ (d-r)\end{array}\right. \text{ by the slope ratio } s.$$

(2) Slip the slope tape A over B with the A graduations running in the $\left.\begin{array}{c}\text{same} \\ \text{opposite}\end{array}\right\}$ direction $\left\{\begin{array}{l}\text{to} \\ \text{as}\end{array}\right.$ those on the B tape. (When $(d-r)$ is negative, as in low banks, the zero of the A tape will be set to $(r-d)$ on the auxiliary tape C).

(3) Assistant stands with reading of tape C, showing half-formation width w on the C.L. Staffman, pulling out the tapes, moves along the cross-section outwards until he comes to a reading on the slope tape A equal to the staff reading called out by the levelman. The foot of the staff will then be at the position for the slope stake, and the reading on the underside of the tape B *plus* the reading of the auxiliary tape C will be the horizontal distance W_s from the centre line to the slope stake.

SETTING-OUT WORKS

Setting out buildings. In setting out buildings, pegs are first put in at the actual sites of the corners, the right angles being commonly set out with the tape, though in the case of extensive or more important buildings the theodolite is used. When the angles to be set out are not right angles, and a theodolite is not available, the corners of the building may be set out by co-ordinate distances, as in Fig. 100, the distances AE, EC and AF, FD being scaled from the plan. As a rule, the close agreement of the measured length of a distant side, such as CD, with its scaled length, will be a sufficient check on the setting-out.

It now remains to reference the positions of the corner pegs, for the latter will be lost when the work of excavating the foundations proceeds; and in this connection the readiest method is to drive pairs of pegs as indicated, so that re-location follows by stretching cords over a pair of pegs in mutually perpendicular lines as in the case of the corner A, which will be found by stretching cords over the pegs 1, 3 and 5, 6. The reference pegs should be well back from the work, clear of all timbering, and should be protected if possible, say, by standing a drain pipe over them.

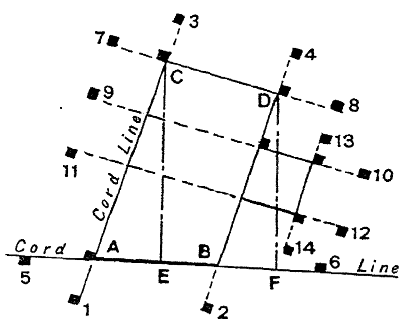


FIG. 100.

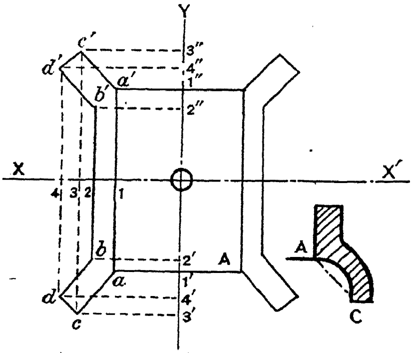


FIG. 101.

Setting out bridge foundations. Doubtless the best method of setting out bridge and culvert foundations is by laying down the co-ordinates of the corners of the abutments and wing walls, the origin being assumed at the centre of the bridge and the axes taken coincident with the centre lines of the railway and the road or stream traversed. The engineer should provide himself with a tracing of the plan of the foundations, as in Fig. 101, and on this he should indicate the co-ordinates of each of the angular points a,a', b,b', c,c', d,d', etc., with reference to the centre line axes XOX' and YOY'; for example, $1a$ and $1'a$ for the point a, $3c$ and $3'c$ for the point c, etc., or the co-ordinate distances 01, 01′, 03, 03′, etc., accordingly. A peg O is driven at the centre of the bridge, and, by means of a theodolite, chaining arrows are inserted in the line XOX' at as many points as may be necessary, so that a cord may be stretched along these. The distances 01, 02, 03, etc., and 01′, 02′, 03′, etc., are measured along their respective lines, and chaining arrows are inserted at the points 1, 2, 3, etc., and 1′, 2′, 3′, etc.

Two tapes are now joined together at their rings, and the chainman, taking this joint, pulls out the tapes while the engineer and his assistant hold these at the required readings on the lines XOX', YOY', as in the case of the point a, which the chainman locates while the engineer and assistant hold the tapes at the arrows 1 and 1′ with respective readings of 01 and 01′. In a similar manner the positions of other points on the foundation are determined and marked with pegs. When all the points have been pegged out, a cord should be passed round the periphery of each abutment, as $acdbb'd'c'a'$, and a cut nicked along this line so that the foreman can make no mistake as to his digging.

When the wing walls are curved, as shown in the detail at A, points in the curve may be set out by offsets to the chord AC, both offsets and distances along the chord being scaled off the plan.

In the case of skew bridges, the procedure is the same, except that the angle between the axes XOX' and YOY' will be set out equal to the angle between the centre lines of the railway, or road and river. Incidentally, it may be mentioned that levels will be taken at the pegs for the dual purpose of estimating the earthwork and giving the foreman figures as to the depth of excavation.

Setting out bridge abutments. When the foundations of a bridge have been laid, the corners of the abutments should be set out accurately with steel tapes, the co-ordinate method being repeated. When one abutment has been built or set out, the corners of the second abutment should be located by measuring from one centre line only, and using the span as measured from the first abutment as the other co-ordinate, thus ensuring correctness of span.

Setting out culverts. Culverts are constructed in so many forms in general practice that it would be impossible here to outline the various problems of setting these out. Arch culverts with wing walls are set out in the manner described for bridges ; and, in general, some modification of the co-ordinate method may be applied to the various types ; pipe, box, old-rail, etc.

Setting out railways. The following are the methods by which the planned location of the centre line of a railway may be transferred to the ground.

(1) Laying down the straights with *reference* to (*a*) mapped objects and points, or (*b*) station pegs of the preliminary survey.

(2) Running the centre line to *alignment* notes prepared as (*a*) polar co-ordinates, or (*b*) rectangular co-ordinates.

(1) **By offset measurements.** (*a*) On this basis, the first method is that used in regard to railways in Great Britain, the offsets being scaled as tie-lines, etc., from the 25″ Ordnance sheets. Here the surveyor begins operations by fixing the ends of the first straight with reference to existing objects, such as fences, buildings, etc., unless, of course, the exact direction is given by some object precisely in line. Having driven the peg at the beginning of the first straight, he will set up the theodolite over this peg, and will sight the range pole at the extremity of the straight, clamping the instrument firmly in this position ; and with the instrument stationed thus, he will direct the driving of pegs to theodolite sights at every chain up to about 15 chains, when it will become necessary to move the instrument forward, in order that the alignment may not be impaired by long sights.

When pegs have been driven to nearly the end of the first straight, the position of the tangent point is scaled, as nearly as may be, from the 25″ map, and the chainmen are directed to stop driving pegs within a chain or so of the estimated position of the tangent point, and to line in arrows instead for a chain or two beyond the estimated position of the point of intersection. The second straight is then fixed by ranging poles near its extremities, the positions of the points defining the straight being tied with distances as scaled from the map. Incidentally, it is well to supplement the map with a complete set of prepared " reference notes ", showing systematically the various objects from which the straights may be tied, their positions checked, and alternatives in the case of awkward or isolated situations.

Now follows the ranging in of the curves, the procedure at the third and succeeding straights being a repetition of the work already described. As in all line work, the chaining will be " through ", or continuous on curve and straight alike from the beginning to the end of the line, pegs being driven at every chain and also at all tangent points and points of junction, regardless of the uneven chainages of such points. The notes as to intersection angles, tangent points, etc., will enable the surveyor to retrace the line during and after construction.

(*b*) In the second method, the points that determine the straights are fixed with reference to the station pegs of the preliminary survey: a method often followed in Colonial work when it is not considered advisable to compile the notes of the methods described hereafter. Thus rectangular offset distances are scaled from convenient stations of the preliminary traverse; and, by laying these off in the field, the directions of the straights are fixed, as in Fig. 102, where the straight *ac* is fixed by the

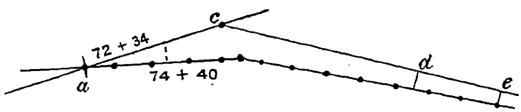


FIG. 102.

intersection at 72·34 and the offset at 74·40, while *ce* may be fixed by the offsets at *d* and *e*. Sometimes an angle is taken off the map with the protractor and set off in the field, making the method similar to the following.

(2) **By alignment data.** Like the preceding method, the following are used in American and Colonial practice; and in general are carried out in connection with a preliminary traverse.

(*a*) **Polar co-ordinates.** In this method the distances between the beginning of the line and successive points of intersection are scaled off and written on the map, each on its proper straight; and the bearing of each straight is taken off with a protractor and indicated duly, the intersection (or deflection) angles being found from the bearings of the intersecting straights. The length of the curve, the tangent lengths, etc., are then calculated with the scaled data and are figured on the map, as also are the " pluses " of the tangent points, etc., together with check points; that is, points where the location centre line intersects the preliminary traverse. When no such intersection occurs, a check line is drawn from a station of the preliminary survey to another on the centre line, and the bearing and length of this line are computed so that it may be run in the field.

The alignment notes are then compiled; usually in the " Bearing-Line " System, columns being provided for *Station*, *Calculated bearing*, *Needle*, *Curve* (containing curve data) and *Vernier* (showing deflection

angles for curves). The pegging out of the centre line is normally carried out in accordance with these notes, and commonly true bearings are used, observations for azimuth being made at specified intervals.

(*b*) **Rectangular co-ordinates.** A more accurate, though more laborious, method is the following, which requires that the preliminary map has been plotted by latitudes and departures (p. 168). Here the tangent points are produced to give the points of intersection; and the co-ordinates of these, together with those of the initial and terminal points, are scaled from the map: the bearing and length of each straight is computed from pairs of successive points, and the values are figured on the map; thus, 13 + 67·3; S. 84° 32′ E. Next the central angle of each curve is found from the bearings of the intersecting straights, the tangent lengths and length of each curve are calculated and figured on the map, and finally the chainages of the tangent points, etc., are carried through and recorded at right angles to the centre line; as, 65 + 9·7, etc. Check data are also obtained, often by the equations of the preliminary and location traverses.

QUESTIONS ON ARTICLE 12

1*. The following staff readings were taken on a line of flying levels between B.M. 73·2 (*High Beach*) and B.M. 68·32 (*Brookland*), a wall and a wide lake of still water occurring in the line.

$\left(\begin{matrix}\textit{High}\\ \textit{Beach}\end{matrix}\right)$ 7·23, $\dfrac{6{\cdot}43}{7{\cdot}21}$, 8·46, (*Wall*) 6·76, $\dfrac{8{\cdot}34}{7{\cdot}86}$, 9 -   ·62, 10·38 (*Brookland*).

The 4th and 5th readings were taken on pegs 11′ 3″ and 11′ 9″ respectively below the top of the wall, while the 8th and 9th were taken on pegs driven flush with the water surface.

Cast up the foregoing notes on a level book form of the Collimation System. (U.L.)

2†. Starting at a B.M. (52·6), 3·08 ft. above the line of collimation, give a continuous sketch showing fictitious staff readings with collimation heights and reduced levels as could occur in running a line of levels against the following difficulties to an assumed closing bench mark:

(*a*) Close boarded fence; (*b*) B.M. 12 ft. above telescope; (*c*) Brick wall, 12 ft. in height; and Lake, 16 chains across.

Also record the notes in a suitable level book. (U.L.)

3†. The following particulars refer to a road 30 ft. in width on a uniform down gradient of 1 in 50, the average normal camber being also 1 in 50. Centre-line distance 1144·8 ft. is the beginning of a circular curve of 250 ft. radius, which at the outer edge is to be banked up to a uniform transverse slope of 1 in 25, the adjustment of camber being made between the centre-line distances 1100 and 1225, keeping the gradient along the outer edge uniform.

Given that the centre-line level at 1100 is 64·82, calculate the levels along the outer edge at each 25 ft. from 1100 to 1225. (U.L.)

4*. The probable error of sighting a target staff with a certain engineer's level is stated to be approximately $0{\cdot}0025 + 0{\cdot}00001(2D - 300)$, the error and sight length D being in ft.

Determine the length of sight at which the combined errors of curvature and refraction are equal to the error of sighting. (U.L.)

[Assuming 8″ per mile for curvature c and refraction r of $\frac{1}{7}c$, $c - r = \frac{4}{7}$ ft. per mile.

$$0{\cdot}25 \times 10^{-2} + 10^{-5}(2D - 300) = \frac{D^2}{(5280)^2} \times \frac{4}{7} = \frac{D^2 \times 10^{-6}}{48{\cdot}787} ;$$

$$D^2 - 975{\cdot}74D + 24393{\cdot}5 = 0 ; \quad D = 950 \text{ ft.}]$$

5†. The error of sighting a target staff with an engineer's level is approximately $E = (D + 0{\cdot}001D^2) \times 10^{-5}$, the error E and sight length D being in feet.

(*a*) Determine the length of sight at which the combined errors of curvature and refraction are equal to the error of sighting.

(*b*) Deduce a rule expressing the probable error between bench marks over a length of M miles, assuming that backsights and foresights averaging 330 ft. are employed. (U.L.)

[$D = 953$ ft. ; $E = 0{\cdot}0124\sqrt{M}$ ft.]

6†. In a constructional survey, two points A and B on the opposite banks of a wide river were located, and, as a check on the levelling work, reciprocal sights from these stations were taken with a theodolite on the target of a staff held on pegs driven flush with the surface of the ground at A and B. The distance AB was known to be exactly 850 ft., and the reciprocal observations were as follows, the target being fixed 10 ft. above the bottom of the staff.

At A, sighting B : Mean vert. angle $+ 1° 28' 20''$; height of axis 4·62 ft.
At B, sighting A : Mean vert. angle $- 0° 52' 40''$; height of axis 4·76 ft.
Determine the elevation of B, given that A was 34·46 feet above datum. (U.L.)

[From A : Mean diff. $A - B = 50{\cdot}93$ ft.
From B : ,, ,, $B - A = 52{\cdot}72$,, Mean, 51·83 ft.]

7*. Four sight rails are to be erected over manhole sites A, B, C, D, 150 ft. apart in a straight line. The invert level of the sewer concerned is to be 90·74 feet at D, being on a gradient of 1 in 150 falling from A to D. Surface pegs are driven at A, B, C, and D, their reduced levels being respectively 101·47, 100·52, 99·74, and 98·67 feet above datum.

Submit the following information : (*a*) suitable length for the boning rod, and (*b*) the corresponding heights of the sight rails above the surface pegs at A, B, C, and D. (U.L.)

[Assuming a sight rail 4′ 3″ high at D :

(*a*) Length of boning rod : $4{\cdot}25 + 98{\cdot}67 - 90{\cdot}74 = 12{\cdot}18$ ft. $= 12' \, 2\frac{1}{4}''$.

(*b*) Height of rail at C : $90{\cdot}74 + 1 + 12{\cdot}18 - 99{\cdot}74 = 4{\cdot}18$ ft. $= 4' \, 2\frac{1}{4}''$.

,, ,, B : $90{\cdot}74 + 2 + 12{\cdot}18 - 100{\cdot}52 = 4{\cdot}40$ ft. $= 4' \, 4\frac{3}{4}''$.

,, ,, A : $90{\cdot}74 + 3 + 12{\cdot}18 - 101{\cdot}47 = 4{\cdot}45$ ft. $= 4' \, 5\frac{1}{4}''$.]

8*. The following are the surface and invert levels relative to a sewer, manholes occurring at the 0′, 200′, and 400′ distances :

Dist. :	0	50	100	150	200	250	300	350	400 ft.
Surface :	97·2	97·8	98·1	98·6	99·4	99·8	100·4	100·8	101·2
Invert :	86·2				87·8				89·4

Plot the longitudinal section and show with dimensioned sketches suitable sight rails at 0, 200, and 400 ft., stating the corresponding length of the boning rod. (U.L.)

9†. The following notes refer to the boning-in of drainage pipes by means of sight rails 150 ft. apart at A, B, C, D, E, and F, the heights of the rails above pegs near the left-hand uprights being as follows :

A	B	C	D	E	F
4′ 5″	4′ 2″	4′ 0½″	4′ 5½″	4′ 5″	4′ 5½″

The invert level of the pipe was 82·16 at A, rising 1 in 150 to F. When the pipes were being laid it was evident that some mistake had occurred, and in consequence the following levels were run on the pegs :

B.M. 91·60	A	B		C	D	E	F	T.B.M.	95·44
3·42	4·78	3·54	3·42	3·76	2·34	1·60	0·76		0·52
			4·36						

Trace the origin of the mistake, and make the necessary correction, submitting your results in tabular form. (U.L.)

[Levels of sight rails at C and D were interchanged.]

10*. The following are the commencement of the notes obtained in running a line with an anallatic tacheometer, the factor, or multiplier, of which was 100.

At Station A : Height of axis, 4·70 ft. ; sighting bench mark, 70·30.

Vertical angle : – 5° 40′ ; cross-wires, 7·58, 5·02, 2·46.

Vertical angle : Sighting Station B. Vertical angle : + 8° 30′ ; cross-wires, 10·40, 6·26, 2·12.

At Station B : Height of axis, 4·55 ft. ; sighting station C.

Vertical angle : + 13° 20′ ; cross-wires, 6·60, 4·22, 1·84.

Record these notes on a suitable tabular form, showing in the proper column the elevations of A, B and C. (U.L.)

[120·93, 240·41, 347·55.]

11*. The following data are extracted from the notes relative to a line levelled tacheometrically with an anallatic tacheometer, the factor, or multiplier, of which was 100.

Instrument station	Height of axis	Staff at	Cross-wires U	Cross-wires M	Cross-wires L	Vertical angle	Remarks
A	4·60	B.M. 70·8	7·60	5·00	2·40	– 6° 30′	B.S. on B.M.
A	4·60	B	7·60	5·20	2·80	+ 6° 30′	F.S. on B.
B	4·50	C	6·90	4·40	1·30	+ 12° 30′	F.S. on C.

Determine the elevations of A, B, and C, and also the horizontal distances AB and BC. (U.L.)

[129·69, 183·08, 301·51 ; 473·83, 533·77 ft.]

12*. Continental surveyors frequently use a supported horizontal staff in conjunction with vertical stadia lines in preference to a staff held perpendicularly to the line of sight.

State the reasons for this, and show how the reductions for inclined sights will differ from those applying to a vertical staff when the latter is used in conjunction with horizontal stadia lines. (U.L.)

13*. The following readings were taken with an anallatic tacheometer, provided with vertical subtense lines to read with a factor of 100 on a 2-metre horizontal staff (H.S.), which was supported with a tripod and carried also a 3-metre vertical staff (V.S.) for use in conjunction with the horizontal line of the diaphragm.

Determine the horizontal distances between A, B, C, and D, also the elevations of these four stations.

Station	Height of axis (m.)	Staff Station	Intercept (H.S.) (m.)	Vertical angle	Horizontal wire (V.S.)		Elevation (m.)
A	1·52	B.M.	1·840	$-5°\ 40'$	1·50		70·645 (B.M.)
		B	1·440	$+3°\ 10'$	1·50		
B	1·44	C	1·760	$+6°\ 15'$	1·50		
C	1·40	D	1·580	$+5°\ 40'$	1·50		

(U.L.)

[143·8, 175·0, 157·2 ; 88·793, 96·768, 115·869, 131·370 metres.]

14*. Describe the principles of that tangential system of tacheometry in which a constant base, say 10 ft., on a vertical staff is employed. Give expressions for both horizontal and vertical distances ; and state your opinions on the method in regard to accuracy, speed, and office calculations, as compared with the stadia, or subtense, system. (U.L.)

[See p. 58]

15*. The following forward and backward bearings were observed in traversing with the compass in a place where local attraction was suspected :

At A : AD, S. 79° 30′ E. ; AB, N. 36° 20′ E.
At B : BA, S. 36° 20′ W. ; BC, S. 65° 40′ E.
At C : CB, N. 66° 20′ W. ; CD, S. 8° 30′ E.
At D : DC, N. 6° 50′ W. ; DA, N. 78° 30′ W.

Tabulate the above readings, and show the *corrected bearings* you would use in plotting the traverse. (U.L.)

[Stations C and D affected ; errors $+40'$ and $-1°\ 00'$.
CD, S. 7° 50′ E. ; DA, N. 79° 30′ W.]

16*. The following notes refer to a compass traverse between two stations A and B, which were fixed by triangulation. Along the course magnetic interference was suspected, and, on this account, both forward and backward bearings were observed.

Given that the magnetic declination was 12° W., plot the traverse with the corrected true bearings on a scale of 200 ft. to 1 inch (using a protractor), and, if necessary, adjust the traverse to fit between the triangulation stations, the latitudes and departures of which are respectively 0, 0 for A and 360 ft. and 1175 ft. for B.

Line	Length (ft.)	Magnetic bearing		True bearing
		Observed	Corrected	
Ab	490	N. 32° E.		
bA	"	S. 32° W.		
bc	175	N. 77° E.		
cb	"	S. 82° W.		
cd	520	S. 68° E.		
dc	"	N. 73° W.		
dB	350	S. 58° E.		
Bd	"	N. 58° W.		

(U.L.)

[Corrected true bearings : N. 20° E. ; N. 65° E. ; N. 85° E. ; S. 70° E.]

17*. Traversing by compass and theodolite may be differentiated as working by "free" and "fixed" needle respectively, the magnetic north and south line being the reference meridian for observed bearings. State systematically the essential difference between traverses made in these ways, particularly noting the effect of errors. (U.L.)

18*. Describe the essential differences between "back angles" and "deflection angles", and summarise the merits and disadvantages of the methods in their application to railway surveying.

Reduce the following deflection angles to magnetic azimuths, the forward magnetic bearing of 232 + 64 from 186 + 65 being N. 74° 20′ E. :

From : 186 + 65, 17° 32′ R. ; 232 + 64, 12° 20′ L. ; 275 + 30, 18° 16′ R. ; 324 + 5, 8° 12′ L. ; and 366 + 30, 14° 42′ L.

Record the reduced azimuths in tabular form. (U.L.).

[74° 20′ ; 62° 00′ ; 80° 16′ ; 72° 04′ ; 57° 22′.]

19*. Discuss the essential differences between "back angles" and "deflection angles", and summarise the merits and demerits of the methods in their application to railway surveying.

Reduce the following back angles to true azimuths, the forward true bearing of 134 + 56 from 88 + 70 being N. 26° 15′ E. :

From 88 + 70, 196° 42′; 134 + 56, 167° 30′; 176 + 24, 198° 24′; 226 + 6, 172° 10′; and 268 + 32, 163° 32′.

Record the true azimuths in tabular form. (U.L.)

[26° 15′; 13° 45′; 32° 9′; 24′ 19′; 7° 51′]

20. Describe in detail:

(*a*) The fixing of points on a plane table by sideways intersection.

(*b*) One method of fixing a plane table station by resection.

(*c*) How to carry out a plane table traverse. (U.C.T.)

21. Define intersection and resection in plane table work. Explain the three-point problem and show how it is solved by Bessel's Solution of the inscribed quadrilateral.

Give a description of the Beaman arc fitted to the telescope of a plane table and explain clearly its uses.

How may contours be obtained with the Indian pattern clinometer? Make a sketch of this instrument. (U.B.)

22. Define "resection" in plane table work. What is meant by the two-point problem? Explain procedure.

Write a note on vertical control in plane table surveys. (U.B.)

23. Discuss the occurrence of the three-point problem in plane table work and describe a solution.

Explain the influence of the relative positions of the points on the accuracy of the fix. In a certain case the position of the table is found to be indeterminate. The reference stations are *A*, *B* and *C*, and the position of the table is at *D*. Line *AC* produced cuts a prominent mark known to be 5 miles distant from *C*. How can *D* be located? (U.G.)

24. Explain the difference between a collimation line and a level line. Discuss the effect the curvature of the earth might have on the results obtained from a line of levelling 5 miles in length, when carried out with an engineer's dumpy level. (I.C.E.)

25. On your examination paper set out a series of contour lines to represent the ridge of a hill through which a cutting, 20 feet wide at bottom, and with side slopes of $1\frac{1}{2}$ horizontal to 1 vertical is to be driven. The contour lines should be at intervals of 20 feet, and be numbered from 300 to 400. Select a central line and taking the elevation at the bottom at 340 feet, draw the intersection of the sides of the cutting with the ground surface so as to illustrate clearly the method you use. A scale of about 50 feet to an inch should be used. (I.C.E.)

26. The difference in level between two points 2,400 feet apart and on opposite banks of an estuary, is determined by reciprocal levelling, using one instrument only. The following readings were obtained:

	Readings on near staff	Readings on distant staff
Level on right bank	4·32	3·34
Level on left bank	4·62	5·88

Calculate the difference in level between the stations and then find the error in the line of collimation of the level, assuming that there is no change in refraction conditions during the period of the observations. The combined error due to refraction and curvature may be taken at 0·02 D^2, where D is the distance between staff stations in thousands of feet, the distance of the level from the nearer staff being negligible. (I.C.E.)

[1·12 ft. true fall ; 0·025 ft. error.]

27. Explain fully, with *neat* sketches, how, in plane tabling, the position of a point may be obtained by resection on two known points. (I.C.E.)

28. What is meant by "magnetic declination"? What is its value in London at the present time? How does this value change during the day and over a long period of time? Define "isogonic lines". In what direction do they cross the British Isles? (I.C.E.)

29. You are sending a subordinate to make a preliminary reconnaissance of several sites for a hydro-electric power scheme. You give him a list of headings under which to tabulate his report. What would these be? (T.C.C.E.)

SECTION III

PHOTOGRAMMETRY

INTRODUCTION

The present section deals with the photographic methods under the headings of ground and aerial surveying.

Ground photographic surveying was first practised by Captain Laussedat of the French Engineers in 1854, although the iconometrical (or perspective) principle had been applied to mapping before the introduction of photography. The possibilities of the method were immediately appreciated in Germany, influencing its progress in Austria, Italy, and later, Russia. Its inception in Canada in 1888 was due to Captain Develle, Surveyor-General of the Dominion Lands ; and from joint operations in Alaska in 1893, the method became a function of the U.S.A. Coast and Geodetic Survey.

Among the notable surveys carried out by the *ordinary* photographic method were Paganini's work in Italian East Africa (1897), Pollack's surveys on the Arlberg Railways (1889), Simon and Koppe's location surveys for the Jungfrau Railway (1891), and Thiele's surveys for the Trans-Siberian Railway (1897). An extended historical note and bibliography will be found in the author's manual, " Phototopography ".*

Although the field work involved was relatively light, which was particularly desirable in difficult or untenable terrain, the method was not altogether economical on account of the consequent office work, particularly in regard to the identification of pictured points. But these difficulties were largely obviated by the stereoscopic principles inherent in the **stereocomparator** of Dr. Karl Pulfrich, of the firm of Carl Zeiss (1902), though a similar device had been described somewhat earlier by H. G. Fourcade, an officer of the South African Forestry Board. New impetus was given to phototopography and ingenious plotting instruments and machines were devised, notably von Orel's **Stereoautograph** and the **Zeiss Stereoplanigraph**, opening up a wide field of possibilities.

Among the best-known applications of the stereophotographic method were Luscher's surveys for the Baghdad Railway (1909), Hebling-Flum's surveys on the Argentine-Chile frontier (1911), surveys in the Karakorum under Prince Louis Amadeus of Savoy (1912), and, after the War of 1914–18, projects in most parts of the world.

* *Phototopography*, A. L. Higgins (Camb. Univ. Press, 1926).

Like aerial surveying, the method has become largely specialised, and the student is referred to a treatise on the subject, such as von Gruber's "Photogrammetry" (trans., 1932).

Aerial photographic surveying may also be attributed to Laussedat, who worked from a captive balloon. Later experiments were carried out by Stolze (1881), Schniffer (1892), while Adams (1893) used the principle of photo-intersections, which has been the basis of the radial line method.

No appreciable progress was made until the outbreak of the War of 1914–18, when the problems of ground control and level flight with level wings at a uniform height were seriously considered. In 1920 Prof. (now Sir) Bennett Melvill Jones and Mr. J. C. Griffiths carried out their well-known experiments on survey flight, and since then much research has been made in the development of flight and technique and in the design of navigational and plotting instruments on the Continent and in America. In Great Britain, much is due to Major Hotine, R.E., who devised the well-known Arundel method, which is characterised by its adherence to simple principles and little apparatus combined with controlled navigation, as opposed to the expensive and elaborate plotting machines so highly favoured on the Continent and to a wide extent in America. It is interesting to note that Dr. Fourcade also introduced (1926) the "correspondence" theory embodied in his **Stereogoniometer.**

It would be out of place in a book of this nature to attempt to describe the numerous instruments that have been evolved since 1920, and for a complete description of these the student is referred to the pamphlets issued by the various makers. Likewise for an intimate knowledge of the subject reference is made to such works as those of Major Hotine, Prof. O. von Gruber, and Dr. C. A. Hart.

Apart from the well-known revision of the 1 : 2500 Ordnance maps of rapidly-developed areas (1925), a large number of surveys has been carried out in connection with railway station sites, revisions under the Town and Country Planning Acts, and surveys for proposed highways, flood prevention, etc., the scales ranging from 1 : 1000 to 1 : 5000, and even as large as 30 ft. to 1 inch. Abroad, the method has been used extensively in preliminary survey for railways, water supply and power schemes, irrigation, and even in connection with high-voltage transmission lines.

The choice between aerial and ground methods will be determined mainly by economical considerations ; and it is impossible to give general comparisons, particularly in regard to a method still in development. In certain surveys, particularly those of highly-developed or difficult areas, the aerial method could be economical, never necessarily cheap ; while in others its cost would exceed that of a ground survey, often at the expense of precision in the details. Its use in special forms of reconnaissance and preliminary survey cannot be questioned, but in larger-scale work it may be an interesting but inexpedient innovation.

Nevertheless, aerial surveying has a wide field, and its possibilities have led to the formation of a number of air survey companies, organised and equipped for all the work of this specialised branch of geodesy.

ARTICLE 1 : PRINCIPLES OF GROUND SURVEY

An outline of ordinary ground photogrammetry will be given in the present article, mainly with reference to *vertical* photographs, which, unlike as in aerial surveying, suggest the plate and not the photographic axis, the latter being horizontal in the present connection.

It has already been stated that a photograph is a perspective view, and in this the " picture plane " is superseded by a vertical photographic plate, the " station point " by a camera Station I, the " distance " by the working focal length f (or distance line) of the lens, and the " centre of vision " by the principal point O, which is the intersection of the optical axis and the plate, O being at the intersection of the horizon and vertical lines (hh and vv) of the photographic perspective. In representing in plan a camera station and the photographs taken thereat, the station is a point and the photograph a line or picture trace pp, which is placed, not behind I as an (inverted) negative, but as an equivalent positive at the corresponding distance in front of the station, the use of such positives being understood in photogrammetry. Also whatever the scale of the map, the trace pp is " oriented " thus in front of the station at the *actual* distance f (or mf in the case of enlargements) ; and thus negatives may be measured in the unit of f (in. or cm.), even though the scale of plotting is in the alternative system.

Now it is impossible to derive a plan (that is, a horizontal orthographic projection) from a single photographic perspective unless the pictured terrain points lie in the horizontal plane, the elevation of which is known with respect to the camera station. Wherefore a geodetic element is introduced in the form of a base between camera stations, and this involves a pair of photographs, each view showing the selected points to be mapped. The process therefore becomes analogous to plane tabling by the method of intersections ; and the treatment may be (1) graphical, or (2) arithmetical.

(1) **Graphical method.** If the stations be plotted in plan as I_0II_0 (Fig. 103) with the picture traces oriented at a perpendicular distance f from I_0 and II_0, then rays from the projection a of a point, pictured as **a** in each photograph, will intersect, giving the plan A_0 of a point A in the

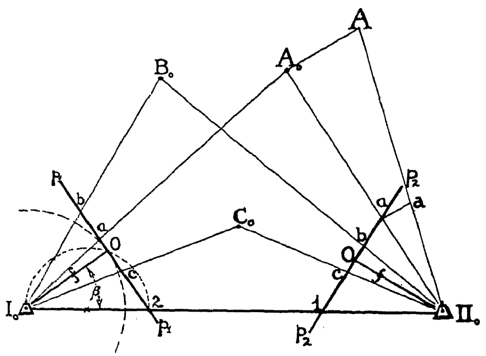


Fig. 103.

terrain*. Likewise for other points, B and C, pictured as **b** and **c** respectively, and projected as b and c, the rays fixing these will intersect at B_0 and C_0 on the plan.

Fig. 104 is a pictorial view showing the equivalent positives in position at the camera stations, the subscripts " 1 " and " 0 " referring respectively to the horizon plane and the datum or ground plane.

Since the horizontal distance II_0A_0 is known, it follows that the difference in elevation H, between A and the camera station II, will be such that:

$$H_1 : \mathbf{a}a_1 : : II_0A_0 :$$

where $\mathbf{a}a_1$ and Oa_1 can be scaled from the photograph as the ordinate y and the abscissa x with respect to the horizon and principal lines accordingly, while II_0a_0 appears on the plotting sheet,

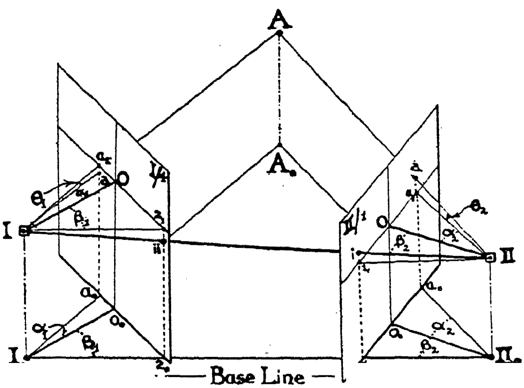


Fig. 104.

Orientation. When the camera stations of a survey have been plotted, the next step consists in orienting the picture traces pp. If a photo-theodolite is used, the angles β between the principal ray

* In the figures of ground photography, points on the photographs are indicated with heavy lowercase letters, the corresponding italics being used for projections, such as a_0, a', a_1, etc.

and a survey line are observed, whereas with a surveying camera it is necessary to picture and plot another station, as II from I and *vice versa*, in Fig. 103. A method of graphical orientation by means of a semicircle is suggested on the left of Fig. 103, but in practice a tee-rule is much more effective. Orientation by means of a photographed compass circle is by no means reliable.

(2) **Arithmetical method.** Let β_1 and β_2 represent the horizontal angles observed at I and II as the angles between the principal line and the opposite camera station, II and I accordingly. Then the data necessary to mapping may be derived from x_1, x_2 and y_1, y_2, the abscissae and ordinates as scaled from the negatives, the focal length f, and the known base length $X = \mathrm{I}_0\mathrm{II}_0$. (Fig. 104.)

(*a*) Horizontal angles α_1 and α_2 between the principal line and the point A as determined at I and II respectively.

$$\tan\alpha_1 = x_1/f\ ;\ \tan\alpha_2 = x_2/f.$$

(*b*) Horizontal distances D_1 and D_2 of the terrene point A, being the sides I_0A_0 and II_0A_0 in the following sine rule relations :

(*c*) Vertical angles θ_1 and θ_2 of the point A with respect to the horizon plane at I and II respectively.

$$\tan\theta_1 = \frac{y_1}{\sqrt{f^2+x_1^2}},\quad \tan\theta_2 = \frac{}{\sqrt{f^2+x_2^2}}.$$

(*d*) Elevation differences H_1 and H_2 between the point A and stations I and II respectively :

$$\tan\theta_1 = \frac{H_1}{D_1} \quad\text{and}\quad H_1 = D_1\tan\theta_1\ ;$$

$$H_2 \quad\text{and}\quad {}_2 = D_2\tan\theta_2$$

*Example**. Describe with reference to sketches the following operations in connection with a surveying camera of the type not fitted with a circle or compass :

(i) Orienting the picture traces with and without a special device.

(ii) Determining the working focal length of the camera. (U.L.)

Here it is necessary that another plotted station shall be pictured, II, say, as *2* from I. (Fig. 105.)

(i) *Graphically.* Describe an arc of radius f with I_0 as centre, cutting I_0II_0 at u. Erect uv equal to $O2$, the abscissa to the picture of II on the photograph taken from I. Describe an arc, also with I_0 as centre, through

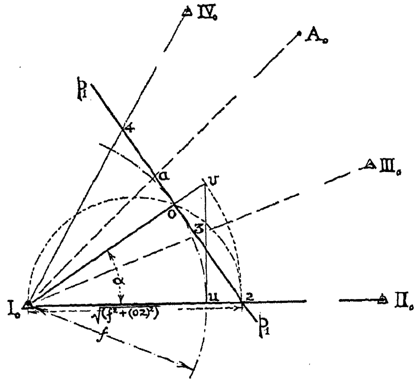


FIG. 105.

v, cutting I_0II_0 in *2*, which is the plan of the pictured point. Describe a semi-circle on $I_0 2$ intersecting the arc of radius f in the required principal point O.

Mechanically. Construct a rule in the shape of a letter "T", preferably of transparent celluloid, making its overall height equal to f, or mf in the case of enlargements. In the centre line of the leg, notch a station centre at the foot, and near the cross-piece cut a triangular hole with an apex for centring on the principal axis.

Measure $O2$ on the edge of the cross-piece equal to the corresponding abscissa on the print; and turn the leg about I_0 until the mark "*2*" falls on the line I_0II_0. Rule the picture trace p_1p_1 along the edge of the cross-piece.

(ii) See p. 75.

*Example**. Draw a neat cross-sectional elevation of a surveying camera or photo-theodolite, indicating the parts peculiar to photographic surveying.

In a photographic survey a point P appears on the negative taken at station A at a distance of 1·80 cm. to the left of the principal line, while on the negative taken at station B, P appears, also to the left, at a distance of 4·70 cm., and 3·20 cm. above the horizon line. The horizontal distance AB is 480 feet, and the photographic axis makes an angle of 63° with AB at A and 56° with BA at B.

Determine the distance AP and the elevation of P, given that the working focus of the camera is 25 cm. and the elevation of the camera axis at B 745 feet above datum. (U.L.)

Refer to Figs. 103 *and* 104 (*page* 212), *writing A, B, and C for I, II, and A respectively.*

$$\text{Tan } \alpha_1 = \frac{1 \cdot 8}{25} = 0 \cdot 072 \; ; \; \alpha_1 = 4° \; 7' \; ; \; \tan \alpha_2 = \frac{4 \cdot 7}{25} = 0 \cdot 188 \; ; \; \alpha_2 = 10° \; 39'.$$

$$\theta_1 + \alpha_1 = 63° + 4° \; 7' = 67° \; 7' \; ; \; PBA = 56° - 10° \; 39' = 45° \; 21'.$$

$$AP = \sin 45° \; 21 \quad \frac{480}{\sin 67° \; 32'} = 369 \cdot 50 \text{ ft.} = D_2.$$

$$BP = \sin 67° \; 7' \; \frac{480}{\sin 67° \; 32'} = 478 \cdot 56 \text{ ft.} = D_2.$$

$$\text{Also} \quad H_2 = \frac{D_2 y_2}{\sqrt{f^2 + x_2^2}} = \frac{478 \cdot 56 \times 3 \cdot 2}{\sqrt{(25)^2 + (4 \cdot 7)^2}} = 60 \cdot 20 \text{ ft.}$$

Reduced level of $P = 745 + 69 \cdot 20 = 805 \cdot 20$.

QUESTIONS ON ARTICLE 1

1*. Describe with reference to sketches the process of orienting picture traces in ordinary photographic surveying :

(*a*) When two triangulation points are pictured, but the principal point is unknown.

(*b*) When three triangulation points are pictured, but both the principal point and the focal length are unknown.

2†. In a photographic survey a point P appears on the negative taken at station A at a distance of $0 \cdot 7''$ to the *left* of the vertical line, while on the negative taken at station B, P appears also to the *left* at a distance of $1 \cdot 80''$ and $1 \cdot 10''$ *above* the horizon line.

The horizontal distance AB is 600 ft., and the camera station makes an angle of 62° with AB at A and 55° with BA at B.

Determine *analytically* the distance AP, also the elevation of P, given that the working focal length of the camera is $5''$ and the elevation of the camera axis at B, 845 ft. above datum. (U.L.)

[358·34 ft. ; 966·0]

3†. Describe Deville's Perspectometer, and explain its use in mapping from ground photographs.

The given print shows a portion of the shore line as photographed when the tide gauges recorded a level of 3·2 ft. below Ordnance datum, the camera station being 11·4 ft. above that datum. Plot the outline of the shore by means of a perspectometer, the focal length of the camera being 6 in. and the scale of plotting 1 : 2500.

4. Describe a photographic theodolite and the method of using it to make a survey. In what kind of country is photographic surveying satisfactory? Explain fully how the survey is plotted and also how contours may be obtained from the photographs. (I.C.E.)

ARTICLE 2: PHOTOGRAMMETRY

The conduct of an ordinary photographic survey consists of field operations and office work, the latter being *photogrammetry* in the strict sense of the term.

FIELD OPERATIONS

The general procedure consists of three parts:

(1) *Reconnaissance*, selecting the triangulation and other camera stations; (2) *triangulation*, surveying these stations; and (3) *camera work*, occupying such stations as will be necessary to the mapping of the terrain.

(1) **Reconnaissance.** (*a*) Occasionally on small scales, such as 1 : 100,000 to 1 : 50,000, the entire work is carried out concurrently with the photography, the surveyor following a certain route with little regard to outlying detail, and completing his operations with progress. In exploratory work of this nature, the camera can be pre-eminent, and a large amount of detail may be surveyed in a very short time by measuring the base lines telemetrically and observing the elevations of controlling points hypsometrically.

(*b*) Certain route surveys are amenable to like treatment, the scales ranging from 1 : 50,000 to 1 : 20,000, including preliminary maps. Greater attention, however, must be given to both instruments and methods in fixing the control points, while the skeleton may become a chain of triangles.

(*c*) Topographical surveys in undeveloped country (1 : 20,000 to 1 : 10,000) and constructional surveys (1 : 10,000 to 1 : 2,500) will ordinarily demand thorough reconnaissance, the object being to select stations which will afford well-conditioned triangles and will lead to definite photo-intersections when serving as camera stations. Maps, if available, should be studied carefully. A sketch map should be the result of the reconnaissance, and areas apparently covered by satisfactory intersections should be indicated on this, reducing the likelihood of any portions being insufficiently surveyed. In extensive surveys, two or more triangulation nets may be essential, the primary net being surveyed before the camera work is begun. The secondary triangulation will be carried out concurrently with the photographic work, the camera stations being fixed in both horizontal and vertical control with reference to the stations of the primary system. Here the sketch map would be superseded by a skeleton map of the entire triangulation and this would enable the surveyor to identify weak points in his scheme and to supplement his camera work accordingly. Frequently a point is occupied and fixed as a camera

station with reference to three pictured triangulation points, the plotted positions of which are known (p. 222). The solution may be (*a*) mechanical, (*b*) geometrical, or (*c*) analytical, though the last should be resorted to only in very accurate work on large scales.

(2) **Triangulation.** The triangulation of an ordinary photographic survey will differ from geodetic triangulation in that clouds would impede photographic work on the loftiest peaks, while otherwise the command would leave much to be desired, as also is the case in flat localities. In extensive surveys which involve a secondary and, possibly, a tertiary triangulation, the primary net is not so much subordinated to the camera work as in the case of a single net with occasional interpolated camera stations. In a simple scheme the net must be amenable to photographic observations, as in the case of the secondary system of an extensive survey. Thus, in general, the triangulation concerned will necessarily be a much finer network than that of an ordinary survey on account of the selection of stations with respect to suitable camera intersections. The camera stations will generally, though not invariably, be triangulation stations also, while, on the other hand, the triangulation stations will not necessarily all be camera stations. The ideal of as few stations as possible consistent with efficient location of detail is to be modified considerably in phototopography, and sufficient stations should always be occupied to obtain full photographic control of the area under survey.

Ordinarily a 5″ or 6″ transit theodolite will meet the requirements of angular measurement, though the angles of a simple or subsidiary system may often be observed with a photo-theodolite concurrently with the camera work. Normally the base line will be measured with the steel tape, the importance of this fundamental measurement being duly observed ; and if the usual precautions are relaxed a number of sides should be measured likewise in order to keep the triangulation under control. As the survey grows in extent and complexity, so must the accuracy of base measurement increase, for here the primary triangulation will detach itself from photography, taking its place as in ordinary surveys. Measurement may be made on the ground if the surface is fairly level and even in smaller surveys, while in extensive schemes a base line apparatus, permanent or improvised, should be used, corrections being applied for sag and slope, also for temperature unless an invar or Konstat tape is employed (p. 340).

(3) **Camera work.** Camera stations should be chosen with due regard to the distances and elevations of the points to be mapped, the focal length of the camera, the requisite degree of accuracy and the proposed scale, and, particularly, the general character of the country. Diversity of surface relief demands diversity of stations, but these should be so distributed that the area can be controlled from as few stations as possible. Judicious occupation of camera stations is a great economic factor, and

advantage should be taken of difficult and elevated stations during good weather and cloud conditions.

Every feature which is to appear on the map must be pictured on at least *two* views taken from two different stations, while it is desirable that all important points shall appear upon three photographs from different stations, in order that a check will be secured. The pictures should overlap so that one or more conspicuous points are common to adjacent prints. Also in order that the pictures may be oriented on the map independently, a triangulation station should appear on each view taken with the surveying camera, though any point that does not contain such a station can still be oriented, provided there is near the edge of the print some point which also appears near the edge of the adjoining print. The number of plates to be exposed at a station will depend upon the field of the camera and the extent of area to be included. Some stations will afford such control that panorama sets will be warranted, six views closing the horizon with adequate overlap, while others will be used merely in order that a single view may be obtained. The limits of a picture will be judged by its image on a ground glass plate, or in the sighting device fitted to the camera; and, in this connection, a sketch should be made showing the limits at each exposure, also any points to which angles are measured. As a rule, one or more vertical and horizontal angles should be taken to well-defined points that will appear in the photographs. The vertical angles may be used in determining elevations, while incidentally checking the position of the horizon line; and the horizontal angles may serve in locating camera stations and in orienting the picture traces, and in giving an incidental check on the focal length and the position of the principal point.

Orientation of the picture traces can be effected when (*a*) horizontal angles are read, (*b*) bearings are observed, or (*c*) a triangulation point otherwise surveyed appears on at least one of the photographs obtained at the station, (*a*) and (*c*) being respectively characteristic of the phototheodolite and the simple surveying camera.

It now remains to consider the exposing of the photographic plates, the instrument having been levelled up to its bubbles in order that the plates are truly vertical. The process will be outlined with reference to plates, though films are used in certain models. Briefly this consists in inserting the plate holder, withdrawing the slide, bringing the collimating notches or lines into contact with the sensitised surface, and making the exposure; then withdrawing the collimating marks from the plate, inserting the slide, and removing the plate holder preparatory to taking another view. All this is more or less the routine of ordinary photography, and is facilitated by the improvements embodied in recent models. On the other hand, a considerable knowledge of photographic technique is essential in the selection of plates, the use of colour screens and diaphragms, and particularly times of exposure.

OFFICE WORK

Apart from the preparation of negatives and prints, this consists of (1) *plotting the triangulation*, (2) *iconometrical plotting*, and (3) *topographical mapping*.

In an extensive survey the entire photographic work should be in charge of an expert photographer, while otherwise the exposures should be made in accordance with his instructions by a member of the survey unit who himself is conversant with the technique of the subject. Phototopography demands far more than amateur experience. On small schemes a member of the unit should be instructed by a professional photographer, who will also prepare the necessary prints and enlargements.

In general, actual size prints can be used when the focal length exceeds 5″ or 120 mm., though enlargements are to be preferred for short focal lengths or otherwise effectively small pictures. Apart from the effects of distortion, enlargements have the advantage in that (*a*) pictured points are more readily identified, (*b*) the accuracy of plotting is enhanced, and (*c*) the picture traces are kept near the edges of the paper, clear of the detail. The plotting sheet seldom admits of a greater magnification than three times, which is the degree of enlargement commonly preferred. Negatives are used in more accurate work by co-ordinate measurement, though diapositives (transparent positives) are used to some extent on the Continent.

(1) **Plotting the triangulation.** The first step consists in plotting the triangulation net, the methods by which this is computed and laid down being determined by the object, extent, and scale of the map.

(*a*) Triangles *primary* to another system would be computed trigonometrically and the net constructed by means of the beam compasses, with due regard to the meridian as determined astronometrically. In work which demands no great precision the net might be constructed by angles with the vernier protractor, as would also be used in the case of a small simple net, though in general co-ordinates are better in the case of single nets. (*b*) Secondary triangles would be adjusted, computed, and constructed in a manner similar to that employed for the primary net, though often co-ordinates may be introduced to advantage. (*c*) Only a very extensive survey would involve a tertiary system in its strict sense, for, as a rule, this would be superseded by camera stations, supplementing the primary and secondary stations. These supplementary stations might be plotted by " chords " though the vernier protractor would be used in many cases, while the tracing paper trammel could be used in cases of " three-point " resection.

(2) **Iconometrical plotting.** This is comprised of three operations: (i) Orienting the picture traces; (ii) identifying picture points; (iii) plotting photo-intersections.

(i) **Orienting the picture traces.** After the triangulation net has been plotted, the picture traces are inserted as lines over part or the whole of the sheet. The process was discussed in the preceding article in regard to the surveying camera and photo-theodolite, in which orientation requires respectively (*a*) one pictured and plotted point, and (*b*) one angle and plotted point. *Successive orientation* consists in inserting a series of picture traces with respect to one which has already been oriented at a given station.

In Case (*a*) it is necessary that there be a point on the right extremity of the first picture which is also on the left extremity of the second picture, or *vice versa*; and so on, the traces being tangential to a circle of radius f—a fact which leads to a simple construction, which may be obviated by the use of a tee rule. In Case (*b*) the angles β will be observed and the principal lines I_0o_1, I_0o_2, etc., set out accordingly without serious regard to the overlap.

(ii) **Identification of points.** In general the area should be divided into sections, and, in the case of extensive surveys, subsections also, the pictures being allocated as far as possible to topographers who have traversed the ground. The photographs should then be examined critically with a view to selecting pairs which exhibit a sufficient number of salient points, each pair, though from a different station, embracing a common area of the terrain.

Some definite system of indexing the photographs and salient points should be adopted in order to avoid confusion. In general, the capital letters might denote primary stations, the small letters substations, while if a survey is divided into sections these could be referenced by the capital roman numerals, the numbers of the photographs at a station being indexed with a subscript numeral, such as A_2, d_3, etc. Arabic numerals should be used for indicating the pictured points selected, red ink being used with a fine dot at the point. The process of identifying points is trying and tedious until the eyes have become trained; and it is this part of the work which has been prejudicial against the wider use of the method. Various artifices are resorted to in the process of identifying and pairing; from the use of large magnifying lenses to stereoscopic comparison where possible.

(iii) **Plotting photo-intersections.** The next stage consists in transferring the abscissae (x) of the indexed points to the picture traces on the plotting sheet. In graphical methods this is best done with the aid of "false traces"—strips of paper on which the principal points are projected, the strip being designated by the index number, etc., by which the photograph is referenced, I_4, say, denoting the fourth picture at Station I. The process can be facilitated by the use of a "transfer board" in which the print (or negative) is placed in a recess while the points are transferred to the false traces by means of a set square bearing

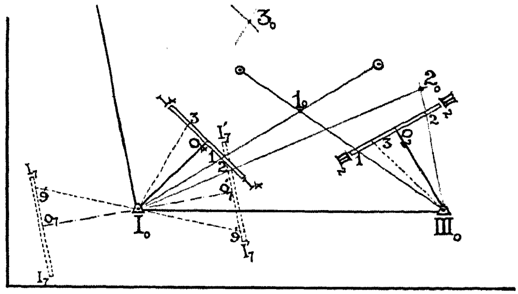


FIG. 106.

against a straight edge fixed parallel to the horizon line. The false traces are then attached to the plotting sheet, as shown in Fig. 106 by I_4I_4 and III_2III_2.

The plan of the point is then determined by the intersections of rays drawn through the corresponding points from the two stations concerned. This process, however, is facilitated by means of silk threads attached to fine rubber bands at one end and at the other to the knobs of 50 gram weights, which keep the threads taut, when the bands are attached to pins at the stations. The process of pricking through the intersections is continued until sufficient points are obtained to determine the outline of the various topographical features and objects.

In the arithmetical method the abscissae and ordinates are measured on the negatives, preferably with a co-ordinatograph, and from these data the angles fixing the direction lines, etc., are determined, as described on page 213.

(3) **Topographical mapping.** The graphical process of determining heights H_1 above the camera axis in accordance with the relation

$$H_1 = \frac{I}{\sqrt{f^2 + x^2}}$$

is shown on the right of Fig. 103. Sometimes the pictured point and its plan A_0 are projected into the principal plane, which is then rabatted about its trace II_0o into the horizontal plane. (Here o replaces O in the figure.) In other words, oa is set off along the picture trace equal to $y = a$a, giving $OA_1 = H_1$ on a perpendicular to II_0o when this axis is produced equal to the principal distance D_0. The consequent relation $H_1 = \frac{D_0 y}{f}$ lends itself to the use of the sector or a scale of heights.

The mapping of contours is not only facilitated in photogrammetry but the delineation of detail is incomparably more natural than if the lines were inserted by memory to mere control points. The work consists in inserting a number of lines with reference to salient points of known

elevation, using the ordinary methods of interpolation and, if necessary, supplementing these with intermediate contours inserted with the aid of the photographs. Incidentally it is advisable to sketch the contours on the photographs with respect to selected control points, since after a while it becomes an easy matter to appreciate the relation between the contours in perspective and their projections in plan.

Although the stereophotographic methods have largely superseded simple phototopography, there are many who prefer simple apparatus and methods combined with the various artifices of perspective, which Deville and others employed with marked success.

Example††. The following notes refer to the interpolation of a camera station P in an ordinary photographic survey with reference to three triangulation stations A, B, and C, which appear on the negative taken at P by means of a camera with a fixed focal length of $6\frac{1}{2}''$.

Determine the co-ordinates of P and its elevation above datum, given that the ground elevation of A was 262·40 above datum, the height of the signal being 10 ft. at this station.

Point	Triangulation co-ordinates		Photo-co-ordinates	
	Lat. (ft.)	Dep. (ft.)	x (ins.)	y (ins.)
A	+1435·0	+ 235·9	−2·82	−0·28
B	+1852·1	+ 617·0	−0·36	
C	+1666·3	+1277·3	+2·65	

The *plus* and *minus* signs indicate points respectively to the right and left of the vertical line and above and below the horizon line when the negative is viewed as a normal erect picture.

From the co-ordinates AB is 565 ft., BC, 686 ft., and CA, 944·5 ft., the bearings of AB and BC respectively N. 42° 24′ 48″ E. and S. 74° 17′ E. and the angle $ABC = \psi = 116° \, 41' \, 48''$.

From the negatives the angles α_1, α_2, and α_3 are :

$$\alpha_1 = \tan^{-1}\frac{2{\cdot}82}{6{\cdot}5}; \qquad \alpha_2 = \quad {}^{-1}\frac{0{\cdot}36}{6{\cdot}5}; \qquad \alpha_3 = \tan^{-1}\frac{2{\cdot}65}{6{\cdot}5}$$

$$= 23° \, 28'. \qquad\qquad = 3° \, 10'. \qquad\qquad = 22° \, 11',$$

giving the angles θ and ϕ, subtended by AB and BC at P as 20° 18′ and 25° 21′ respectively.

Adopting the notation of Fig. 83, page 163, with α and γ the unknown angles of A and C respectively, $\alpha + \gamma = \delta$, where

$$\delta = 360° \, 0' - 20° \, 18' - 25° \, 21' - 116° \, 41' \, 48'' = 197° \, 39' \, 12''.$$

Substituting in $\cot \alpha = \dfrac{c \sin \phi}{\sin \theta \sin \delta} + \cot \delta$, where c and a are the sides subtending the angles at C and A in the triangle ABC :

$\alpha = 101° \, 49' \, 20''$, giving the bearing of AP, S. 35° 45′ 32″ E.

Also $\quad \dfrac{c \sin ABP}{\Delta}$, where $ABP = 57° \, 52' \, 40''$; $AP = 1379{\cdot}2$ ft.

Whence lat. of $P = -1119{\cdot}2$ ft. and dep. of $P = +806{\cdot}1$, giving total co-ordinates of 315·8 N. and 1042·9 E. respectively.

$\text{Tan } \theta_1 = \dfrac{y}{} \quad \dfrac{-0{\cdot}28}{\sqrt{(6{\cdot}5)^2 + (2{\cdot}82)^2}} = -0{\cdot}03952$, where θ_1 is the vertical angle. Hence the depth of A below the camera station P is

$$H_1 = AP \tan \theta_1 = 54{\cdot}51 \text{ ft.},$$

giving the elevation of the camera axis at P 326·91 ft. above datum.

QUESTIONS ON ARTICLE 2

1†. Describe concisely how you would conduct a topographical survey of about four square miles in area by the ordinary photographic method in open mountainous country.

State the equipment you would require and how you would organise your survey unit, treating your discussion under the following headings :

(*a*) Triangulation in field and office ; (*b*) Mapping topographical details ; (*c*) Inserting contour lines.

2†. Describe with sketches the field work of a survey with a photo-theodolite. How would you plot the survey?

What special difficulties arise in aerial surveying which do not arise in ground survey? (U.L.)

3†. A survey is to be made of an island about 10 sq. miles in area by the ordinary photographic method, the country being difficult but open.

Draw up a scheme for the survey, stating the equipment and personnel you would require, and describe concisely how you would conduct the survey, outlining all essential operations. (U.L.)

4*. Describe with reference to a sketch Deville's method of Vertical Intersections, explaining its use in ordinary phototopography.

5†. Describe concisely with reference to diagrams the methods of working with inclined photographs, giving particular attention to the following :

(*a*) Inserting the picture traces ; (*b*) Plotting direction lines ; (*c*) Determining elevations of pictured points.

6. In a survey carried out with a photo-theodolite, the length of the base AB between two of the stations was 1,000 feet. C was a station outside the line AB. In the photograph taken at A with the instrument sighted to C

and with angle $CAB = 33° \ 10'$, the mast of a wireless station appeared 1 inch to the left of the principal vertical line in the print. In the photograph taken from B, with the instrument again sighted to C, and the angle $CBA = 56° \ 48'$, the same mast appeared $\frac{3}{4}$ inch to the left of the principal vertical line. Calculate the distance of the mast from C. The focal length of the photo-theodolite was 6 inches. (I.C.E.)

[151·0 ft.]

ARTICLE 3: STEREOPHOTOGRAMMETRY

Stereophotogrammetry, as the word implies, introduces the function of stereoscopic impression in the process of mapping, utilising stereoscopic images, or plastics, instead of photographic perspectives.

The geodetic element, virtually a measured extension of the eye-base, or epipolar axis, also assumes a physiological function in the reconstruction of the stereoscopic relief, while through the medium of a stereocomparator, the depth of impression gives the measurement of another element, the parallax p, which with the usual x and y co-ordinates supplies the third dimension essential to the location of a point in space.

STEREOSCOPIC IMPRESSION

Stereoscopic impression is that complex function of the eye, by virtue of which relative distances are gauged subconsciously in binocular vision. It is the effect produced by the variation in position of images on the retinas of the eyes when rays are received from objects at different distances, and thus a plastic relief is seen and not merely a perspective view on a plane surface. Although certain geometrical rules may be enunciated, these are influenced by uncertain physiological factors, giving elasticity, which is a great asset in practice, as the interpretation of aerial photographs has shown.

If a point P on the perpendicular at the midpoint of the eye-base b is viewed, the rays will make equal angles γ with the zero axes of the eyes, while if a nearer point Q on the same perpendicular is sighted, the angles will be δ accordingly, the difference $\phi = \delta - \gamma$ being the parallactic angle, and the distance PQ the " stereoscopic depth ". Now if the eye-base b, or interocular distance, which is about 65 mm., be increased n times by mirrors or prisms, as in Helmholtz' telestereoscope, the appreciation of parallactic differences will be increased accordingly, giving a base $B = nb$, as in Fig. 107, where the eyes at oo will see the landscape as though they were at ee.

Further, if the arrangement in Fig. 107 were combined with binoculars, the angles ϕ would be increased in the ratio m, so that the total increase in stereoscopic effect would be theoretically mn, which is known as the "total plastic" of the arrangement. On the other hand, however, the increase in the angles ϕ by magnification would mean a reduction in stereoscopic depth, as would be evident from the geometry of the figure suggested in a preceding paragraph. In practice, therefore, the selection of the ratio n/m is a matter of great importance in the design of instruments, which also include stereotelemeters and measuring stereoscopes.

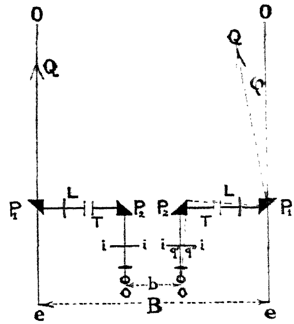


FIG. 107.

The stereocomparator. In stereophotogrammetry the space relief of the landscape is reconstructed in the stereocomparator from two photographs, which, in the common case, are taken from the ends of a short base line with both plates in a vertical plane parallel to the base line. At the image planes of the binocular microscopes are two image plates, which, replacing the aerial plates of range-finding binoculars, carry a fixed vertical index line, and these lines when merged, or fused stereoscopically, become a single "wandering mark", or as the index is styled in aerial survey, a "floating mark", since it then appears to float rather than wander.

Scales are provided for the measurement of the essential co-ordinates x_1, y_1 and p, the subscript "1" referring to the left-hand pictures of the pairs, which may be negatives, diapositives, or even prints in certain types of comparators.

The principles of stereocomparison will be evident from Fig. 108, which though merely diagrammatic, is substantially the same as the stereomicrometer with which Dr. Pulfrich demonstrated the manipulation of the stereocomparator.

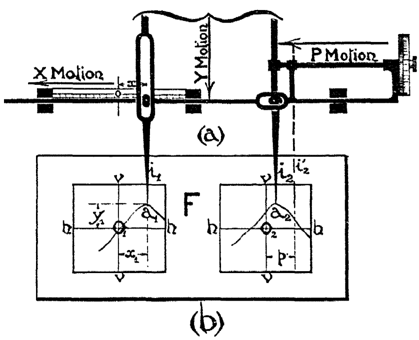


FIG. 108.

Fig. 108 (b) is a frame carrying a stereoscopic pair, each showing the principal and horizon lines, vv and hh respectively. Fig. 108 (a) is a micrometer which carries two fine pointers i_1 and i_2 over the pictures and embodies scales and motions which

control the movements of the pointers ; viz., X, showing the abscissa x_1 to i_1 ; Y, the ordinate y_1 to i_1, and P a function of the parallax ϕ to i_2.

If the pointers be set to the principal points O_1 and O_2, the scale readings will give the zeroes for X, Y, and P ; and the pointers in binocular vision will merge into a single pointer indefinitely distant in space. Next, if the pointer be set to the picture a_1, as seen only through the left eyepiece, the readings on X and Y will be respectively the x and y coordinates of a, the pointer i_2 coming to the position i_2'. Finally, if the pictures be viewed stereoscopically, and the fused wandering mark i be made to emerge through the relief until it coincides with the point a, the movement p of P will be known, being the linear parallax. Normally these motions are modified in four respects in a stereocomparator.

At the present time various models of stereocomparators are procurable, apart from those more readily adapted to aerial survey, such as the Thompson, or Cambridge Comparator. Except for improvements and refinements for specific purposes, the essential features are the same, and may therefore be described with reference to an earlier Zeiss model (Fig. 109).

This instrument consists of two primary portions, which may be styled (1) *the measuring table*, and (2) *the stereo-head.*

(1) The base, or bed, is a cast table A fitted with four levelling screws, a, a, as feet. It carries on its surface a horizontal sliding platform F,

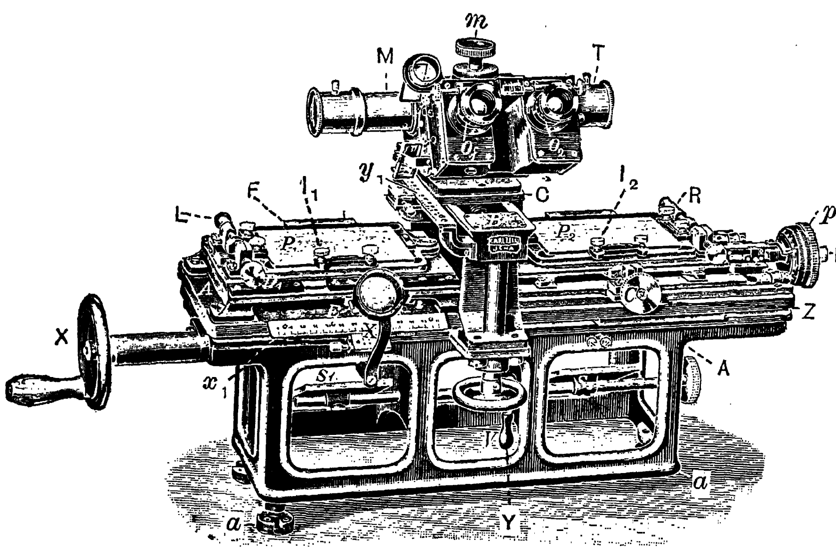


FIG. 109. Zeiss stereocomparator

which is provided with recesses and clamps R and L for securing the photographic negatives, I_1 and I_2. In addition it supports the binocular microscope M and its attachments, and provides bearings for the remaining motions and adjustments, P, X, Z, etc.

(i) The *breadth* motion consists of a screw operated by a hand-wheel X, by means of which the negatives are moved equally and simultaneously, the amount of movement, the abscissa, being shown for the left-hand plate at a vernier index on the scale x_1.

(ii) The *depth* motion consists of a micrometer screw, which moves the right-hand negative independently along the X axis, the amount of movement, the *linear parallax*, being indicated on the scale p of the head P.

(iii) The *transverse* adjustment consists of a screw Z which moves the carrier of the right-hand negative independently, and so allows the pictured point to be brought into the same horizontal plane, as it were setting the photographs to a common datum in the case of plates exposed at different elevations. A refinement of this motion is sometimes referred to as vertical parallax motion, the depth motion being that of horizontal parallax.

(2) The *stereo-head* consists of (a) the *telestereoscope*, and (b) its *carrier stand.*

(a) The telestereoscope T is comprised substantially of all the components of Fig. 107, a system of Porro prisms erecting the images viewed with the binocular microscope M, which gives an enlargement from 3 to 6 diameters in the model shown. The pointers, or indexes, each consists of a vertical line on a thin glass plate i, the latter being set in the image planes of the microscopes. Each eyepiece o_1, o_2 is independently adjustable to the eye, while the entire microscope M may be raised or lowered by means of the screw m.

(b) The carrier standard C consists of a slide-rest arrangement fixed to the centre of the table A. It serves as a standard for supporting the telestereoscope T and embodies the motion with which the latter is moved perpendicularly to the movement of the platform F.

(c) The height motion consists of a rack and pinion operated by the hand-wheel Y, which moves the telestereoscope across the table, the amount of movement, the *ordinate*, being shown for the left-hand picture at a vernier index on the scale y_1.

The main x and y co-ordinates are read by a vernier to 0·02 mm., and the parallax to 0·01 mm. by micrometer, a like refinement replacing the Z motion in more recent models. Estimate of parallax is thus possible to 0·001 mm., though the ultimate accuracy would be to about 0·01 mm.

The *modus operandi* of determining x_1, y_1, and p for a plate pair is as follows : (1) Secure the negatives I_1, I_2, as erect pictures in the platform

recesses R and L with their horizon lines parallel to the X axis. (2) Adjust the negatives so that their principal points O_1, O_2 are coincident with the principal rays, using the hand-wheels X and Y, the micrometer head P, and if necessary the screw Z. Read the scales x_1, y_1, and p for the initial readings, which all should be zero in a perfectly adjusted instrument. (3) Select a point a in the plastic, and, by means of the hand-wheels X and Y, bring the wandering mark to this point, setting it to its position in the relief by means of the head P. (4) Read and record the data, completely locating the point A ; viz., x_1, y_1, and p.

STEREOSCOPIC SURVEYING

In general, a photo-theodolite will be essential in order that the plate pairs may normally be exposed in a vertical plane, coincident with, or parallel to, the stereoscopic base, while the contingency of inclined plates renders a vertical arc desirable. Usually the photo-theodolite and stereocomparator are constructed to common dimensions, a complete outfit being supplied by certain makers. Some comparators, however, are independent of the camera, provided the plates are smaller than the maximum dimensions.

As in ordinary photographic surveying, the field work will consist of the following:

(1) **Reconnaissance.** Selecting the triangulation stations and such additional camera stations as will be essential to the efficient location of detail.

(2) **Triangulation.** The triangulation will not differ appreciably from that of ordinary trigonometrical surveys, except to some extent in the minor or tertiary triangles, for in stereophototopography each camera station has its own substation at the other extremity of the base, and is not part of a system as in the case of ordinary photographic surveying. The trigonometrical work is carried out in the usual manner, locating the stations both in horizontal and vertical control.

(3) **Camera work.** The actual photography is carried out as in the ordinary method and therefore need not be reconsidered.

The chief consideration is the determination of the necessary stereoscopic bases, consistent with the limits of error prescribed for the survey, both in horizontal and vertical control. The lengths of these bases are limited by the fact that plate pairs cannot be viewed in plastic to their full extent when the bases exceed a certain length. The limits of stereoscopic vision have been critically investigated in recent years. Pictures derived from an inordinately long base will have zones that alone admit of stereoscopic examination, and areas outside these, both nearer and farther, will appear blurred and indistinct, necessitating re-focussing for every change of distance.

Now, by analogy with Fig. 110, it will be seen that the horizontal

principal distance D of a point from the left-hand station of a base of length B is such that

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where f is the focal length of the camera and p the parallax as read on the scale of the comparator. Differentiating,

$$dD = f\frac{B}{p^2}\,dp. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

But
$$p = \frac{fB}{D}\,;\ \text{and hence}\ dD = \frac{D^2}{Bf}\cdot dp. \quad \ldots\ldots \quad (3)$$

Also if H_1 is the height of a point above the camera axis,

$$H_1 = \frac{Dy_1}{f} = \frac{By_1}{p}\,, \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

where y is the ordinate of the pictured point as measured on the left-hand negative.

Likewise
$$dH = \frac{D}{f}\cdot dy + \frac{y}{f}\,dD, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

since the square root of the squares of the errors could not be taken defensibly in the present connection.

Hence the base will be calculated from

$$B = \frac{D^2\,.\,dp}{f\,.\,dD}\,, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

with reasonably assumed values for the errors in parallax and horizontal distances, dp and dD respectively.

PHOTOGRAMMETRY

Some cartographers plot horizontal distances L directly along the direction lines from the left-hand stations, while others use the corresponding principal distance D, or lengths projected on to the principal axis, which are D_s to scale in Fig. 110, where p_1p_1 is the picture trace oriented at the left-hand station, I_0 in plan.

Here the abscissa x_1 is set off in the ordinary way from the principal point o_1, giving the direction line I_0a to the pictured point A_0. The principal distance D follows from $D = fB/p$, and is set off to scale as D_s along the principal ray as I_0A_1, the projector from A_1 intersecting the direction line in the required point A_0.

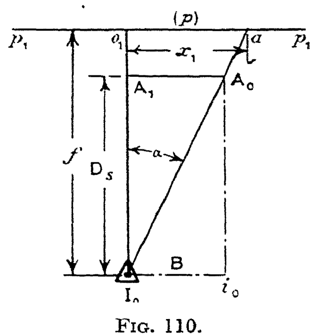


FIG. 110.

Greater accuracy follows from plotting radial distances $L=\sqrt{D^2+x_1^2}$ preferably with false picture traces at a distance mf and abscissae also enlarged to mx, m being $1\frac{1}{2}$ to 2 conveniently. Normally the product fB is constant, and this suggests the use of diagrams, one of which is indicated in dotted lines, with B set off to scale and x_1 replaced by the parallax.

Elevations above or below the horizon line of the left-hand stations are determined by the relation given in (4), which also may be applied graphically or instrumentally.

Contours. It follows from the equations for elevations that the wandering mark will move in a frontal plane—parallel to the picture plane—when no change is made in p, and consequently in D, f and B being normally constant. Thus a longitudinal section, or profile, may be taken in any frontal plane by merely varying y_1 with the Y motion. Also with y_1 fixed for any particular contour, all points touched with the wandering mark will be on that contour ; and if p be varied and altered accordingly H_1 may still have the same value, and contour points may be located in frontal planes as determined by the factor By_1/p. Further, if the wandering mark be localised to a given terrene feature, and H_1 be kept constant by varying y_1 and p, the trend of any particular contour may be quickly determined, and retraced by the observed values of p in $D=fB/p$, the wandering mark becoming as a staffman in direct contour location.

Automatic plotting machines. Although ingenious diagrams and drafting devices were constructed to facilitate the application of the distance and elevation formulae of stereophotogrammetry, the extended use of the method soon led to the development of mechanical plotters with which it was possible to obviate the reading of scales and the consequent reductions, the survey being plotted automatically, even to the tracing of the contours.

Only brief reference to these machines is possible in a work of this nature, since stereophotogrammetry is now a specialist branch of geodesy, and appropriate description would demand a treatise alone. Among the instruments that may be mentioned is the well-known Zeiss Stereoplanigraph, which is a universal instrument, adapted to the mapping of both aerial and ground surveys, each on different scales simultaneously, without regard to the relative positions of the photographic plates at the time of exposure, the relative orientation of the plates in space being actually reproduced. The Wild Autograph, another instrument adapted to mapping alike from aerial and ground photographs, is used extensively on the Continent and in America. The instrument that opened up the stereophotographic method to automatic plotting was styled the Stereoautograph by its inventor, Major von Orel, of the Austrian Geographical Institute. Incidentally, a year earlier, in 1907, Major F. V. Thompson, R.E., devised a stereoplotter, which was almost automatic.

The stereoautograph. Fundamentally, this instrument transforms the distance relation $D = Bf/p$ into $D : kB :: f/k : p$, in order to determine practicable intersections by increasing the intersection angle in the same ratio as the base line.

The instrument consists of a table in which the horizontal frame carries the three primary systems, essentially, (1) *the modified stereo-comparator*, (2) the *drafting mechanism*, and, as an accessory, (3) the *plotting board.*

(1) The comparator differs from the independent instrument already described in that normally no parallax, or P, scale is involved with regard to the right-hand negative I_2, the P motion being directly communicated to the drafting mechanism. Otherwise the sliding platform F and microscope M are utilised, while a recess is provided for a photographic print on the left of the negatives I_1 and I_2, the print having the motions of the left-hand plate. Also advantage is taken of the sliding motions in that points in these shall form centres or pivots for the lever arms of the autograph (Fig. 111, p. 232).

(2) The drafting mechanism consists of the gears, slides, and levers which transmit the X, Y, and P motions to the plotting pencil A_0. In addition to the various clamps and fine adjustments, the mechanism consists essentially of (a) the X, or *breadth* motion, which works with reference to the left-hand plate of the comparator, and so transforms the horizontal co-ordinate x_1 to the corresponding angular position; (b) the Y, or *height* motion, which determines the ordinate y_1 from the left-hand plate, transforming this by the relation $H_1 = \frac{By_1}{p}$, and affording elevations H which may be (i) read independently as H_1 with reference to the left-hand station, or as H_0 above datum, or (ii) embodied as an erstwhile fixed elevation, determining contours in plan and also delineating them in perspective on the photographic print; (c) the P or *depth* motion, which moves the right-hand negative and simultaneously transposes the consequent parallax so that the corresponding distance is fixed mechanically along the direction determined by the X motion.

(3) The plotting board is an adjunct provided with a plotting point or pencil A_0; and in this connection highly-developed methods and craftsmanship are combined in eliminating the numerous errors that may easily arise even from well-constructed mechanism.

Fig. 111 is a diagrammatic plan of the stereoautograph, and, being drawn solely to illustrate the application of fundamental principles various details are necessarily omitted and certain parts of the mechanism incorrectly disposed.

Here, three levers X_1x_0, Y_1y_0, and P_1p_0 are pivoted at centres X_0, Y_0, and P_0, so that the points x_0, y_0, and p_0 can transmit the motions

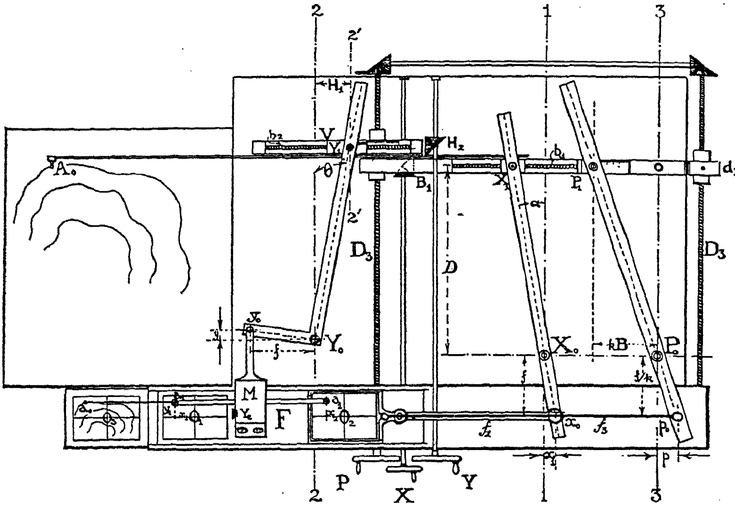


FIG. 111. Diagrammatic plan of stereoautograph

corresponding respectively to X, Y, and P to the separate slide rods f_1, f_3 and the microscope M, the points X_1, Y_1, and P_1 themselves giving magnifications of the x and y co-ordinates and a function of the parallax p. The distance f represents the constant *focal length*, and D the *principal distance* to scale, a dimension depending upon the selected scale of the map.

In Fig. 111 the platform F is shown with reference to the microscope M, so that an abscissa x_1 and an ordinate y_1 determine a point a on the left-hand plate I_1. The movement of the slide for the abscissa x_1 is transmitted to the pivot x_0, giving motion accordingly to the point X_1 of the lever X_1x_0. Thus, with the breadth wheel X (in conjunction with Y) the index of the microscope is brought to the point $\mathbf{a}_1$ on the plate I_1; and by the movement x_1 the corresponding angle α is virtually set out from the centre line 1/1, α being the horizontal angle between a perpendicular to the base line at the left-hand station and a terrain point A, which appears as $\mathbf{a}_1$ on the left-hand plate. Conversely the motion of the microscope to the ordinate y_1 is received at the pivot y_0, following the displacement of the crank and lever arm Y_0Y_1, which latter at Y_1 receives the movement of the screw h_2 as transmitted through the pinions H_2 from the height wheel Y. Hence the arm Y_0Y_1 describes the vertical angle θ between the point A and a horizontal plane through the axis of the camera at the left-hand station, the trace of this plane being represented by 2/2. Thus the index i_1 is set by the X and Y motions to a point $\mathbf{a}_1$ on the left-hand plate, while the mechanism reproduces the

respective horizontal and vertical angles α and θ to the corresponding terrain point A. But the parallax p must be introduced before the horizontal distance D and vertical intercept H_1 are determinate.

Now a straight edge X_1P_1 lies parallel to the X axis, and is moved perpendicularly (i.e. in " depth ") across the table by means of the carriers d_3d_3 which serve as nuts to a pair of worm screws D_3D_3. These screws are operated simultaneously by means of the depth wheel P, and the displacement affords the parallax movement at p_0, thus bringing the index i to the point a as seen in the stereoscopic relief. Actually, however, the right-hand negative is moved in accordance with the function of the parallax reproduced at P_1.

Thus the motion of P determines the parallax and principal distance simultaneously, whereas hitherto the distance was calculated to correspond with a predetermined parallax. In fact the partial solution in Fig. 110 is performed mechanically, while the intersection of the line of the arm X_0X_1 with the upper horizontal line of the figure determines the position of the point A in plan, introducing the partial solution of Fig. 110. The point A is merely transferred to A_0 on the plotting board. It is, therefore, evident that the line of the straight edge X_1P_1 must contain the horizontal projections of all points of the same parallax; that is, $fB/p=D$ is constant; and since B and f are normally constant, a change in D must accompany a change in the parallax p. However, the effective base kB may be varied to admit of a change in the scale of the survey, which is limited by the size of the board and extent of terrain to be mapped.

In ordinary surveys the Y motion is utilised only in so far as it is necessary to bring the index to representative points in the plates, the mere plan following from the directions α and distances D, which are respectively functions of the abscissae x_1 and the parallaxes p.

Nevertheless the vertical angles θ may be transformed into the corresponding vertical intercepts H_1 by the introduction of the parallax function in accordance with $H_1=By_1/p$; and to this end an elevation scale shown at V makes it possible to read elevations (a) H_1 above instrument, or (b) above datum H_0, the latter by simply setting the scale at V to show the reduced elevation of the camera stations. In particular, it is possible to determine and trace contour lines automatically by merely coupling the Y motion to the already combined X and P motions. This important topographical operation is effected by first setting the height scale to a division corresponding to the scale of the map, and then setting the clamp at Y_1 so that H_1 accords with the required contour elevation, the movement of the microscope being otherwise released by loosening the clamp Y_2. Then, if the wandering mark is made to touch the plastic picture by means of the X and P motions, it will locate points of equal altitude, while the straight edge X_1P_1 will follow the line $2'/2'$, which is

really the contour plane corresponding to the elevation determined by H_1. Normally the trend of a contour is followed from the left of the pictures, and the operation is effected without removing the eyes from the microscope, the carriage of which co-ordinately follows the course of the wandering mark. At the same time it is possible to insert the perspectives of the contours on a photographic print I_3 which is fixed in the extreme left of the slide F; for here it is necessary only to transmit to the print the x_1 and y_1 movements corresponding to the motion of the index on the left-hand plate, the former following the motion of the slide F and the latter that of the microscope, as transferred to the pencil at $\mathbf{a}_0$. The accuracy with which contours are inserted depends almost entirely upon the ability of the operator to see stereoscopically, and to acquire an almost instinctive touch in making the wandering mark follow the surface of the plastic model.

The autograph has been developed in different patterns by certain firms, one Zeiss model employing a pedal movement for actuating the height motion Y, and thus permitting the simultaneous use of all three motions.

The advantages of the stereophotographic method over the ordinary method of photographic surveying, may be stated concisely, as follows:

(1) Facility of identifying points, which, at best, is trying and difficult in ordinary photogrammetry. Besides, the observer sees the landscape as he sees it in Nature, and not as a mere perspective picture. (2) Also the method is not restricted to surveys in clear mountainous country, and dense vegetation is no longer a serious impediment. Besides, features on comparatively level ground are as readily and accurately located as conspicuous points. Nor need the photographs be necessarily sharp, for exposures in poor light and weak positions afford acceptable results in the stereocomparator. (3) Photo-intersections are obviated, and the triangulation is no longer subordinated to the choice of camera stations. Thus the triangulation system is simplified and the camera stations are substantially fewer, the field and office work being reduced accordingly. (4) Further, the accuracy of the photographic method is raised to the standard of good tacheometrical surveys. For instance, in railway location surveys with average distances of 200 to 400 metres, measurements may be made to within 0·2 to 0·3 metre for horizontal distances and 0·07 to 0·11 metre for elevations. (5) Finally, the combined field and office work is very considerably shorter than that of tacheometrical surveys generally, while the resulting contour maps contain much more detail. Also the office work may be reduced to simple routine by the use of the stereoautograph, which practically halves the labour of ordinary comparator work, even with the aid of a plotting diagram. The stereoplanigraph opens up a wide field of cartographic reproduction.

Example†. Determine the horizontal distances and elevations of the points P, Q, and R from the following stereocomparator notes which refer to ground observations with a 50-metre base and a photo-theodolite with a working focal length of 150 mm., the camera axis being horizontal at a station O.

Height of camera axis, 1·40 m. Elevation of O, 210·36 m.

Point	Co-ordinates		Parallax
	x	y	
P	−4·40	+2·40	2·10
Q	+1·70	−0·30	1·80
R	+2·30	+1·40	1·50

In the above notes x and y are respectively the abscissae and ordinates of the left-hand negatives with respect to an origin at the principal points, all stereocomparator readings being in cm. (U.L.)

Principal Dists. $D=\dfrac{Bf}{p}=\dfrac{7\cdot5}{p}$ m., while the actual horizontal distance $L=D\dfrac{\sqrt{f^2+x^2}}{f}=D\sec\alpha$, α being the horizontal angle between the principal and direction lines. Also the heights H_1 above or below the camera axis are $H_1=\dfrac{Dy}{f}=\dfrac{By}{p}$, being determinate from the rectangular measurements, since

$$H_1=\frac{Ly}{\sqrt{f^2+x^2}}=\frac{D\sec\alpha}{f\sec\alpha}\cdot y. \qquad \left[\operatorname{Tan}\alpha=\frac{x}{f}\right].$$

Horizontal Dists. (m.) 372·18, 419·36, 505·82 ; Elevs. (m.) 268·90, 203·43, 258·42.

Example†. A ground stereophotographic survey was carried out with vertical plate pairs from a 50 metre horizontal base with a 13 cm. × 18 cm. outfit, the focal length being 150 mm.

Tabulate for unit values of the y ordinate the corresponding parallaxes for tracing contours at 10 m. intervals between 100 and 150 m. by means of the stereocomparator with reference to a station at which the camera horizon was 116·20 m. above datum. Insert also the corresponding principal distances to the contour points.

Let y_1 be the ordinate, p the parallax, f the focal length, D the principal distance, B the base, and H_1 the height of a given contour above (or below) the camera axis : then

$$y_1=\frac{H_1}{B}\cdot p=\frac{H_1}{D}\cdot f.$$

Thus, for the 110 contour,

$$H_1 = 116{\cdot}2 - 110 = 6{\cdot}20,$$

$$p = \frac{50}{6{\cdot}20}\, y_1 = 8{\cdot}06 \times 1 \text{ cm.}; \quad D = \frac{50 \times 15}{p} = 93{\cdot}1 \text{ m.}$$

Contour :	100	110	120	130	140	150
Parallax :	3·09	8·06	13·16	3·62	2·10	1·48 cm.
Prin. Dist. :	242·7	93·1	56·9	207·2	357·1	506·7 m.

QUESTIONS ON ARTICLE 3

1†. The following stereocomparator notes refer to a photogrammetric survey in which a camera having a focal length of 240 mm. was employed, the base, or station line, being consistently 50 m. in length. Height of Camera Axis, 1·34 m.; Elevation of Station, 125 m. above datum.

Point	1	2	3	4
Abscissa (cm.) - -	− 4·22	− 1·66	+ 2·34	+ 3·73
Ordinate (cm.) - -	+ 2·34	+ 0·28	+ 1·12	+ 0·94
Parallax (cm.) - -	0·60	1·10	0·90	1·40

Determine the horizontal distances and elevations above datum of the points 1, 2, 3, and 4 on the plate pairs. (U.L.)

[Hor. Dists. : 2025, 1094, 1339, 867 m., Elevs. : 321·3, 139·1, 188·6, 159·9 m.]

2†. A survey is to be made of an island of about 10 square miles by the stereophotographic method, the country being difficult and fairly closely wooded.

Draw up a scheme for the survey, stating the equipment you would require; and describe concisely how you would conduct the survey, outlining all essential operations. (U.L.)

3†. A preliminary survey for a railway is to be carried out by the stereophotographic method in difficult country.

Draw up a scheme for the survey, stating the equipment you would require; and describe concisely how you would conduct the survey, outlining in particular the following operations: (*a*) Determining length of base for the plate pairs, (*b*) Mapping important objects, and (*c*) Determining points on the contours. (U.L.)

[Assuming errors in measurements on the negatives of 0·05 mm., and 0·5 m. error in distance up to 500 m.,

Base $B = \dfrac{D^2 \,.\, \Delta p}{f \,.\, \Delta D} = \dfrac{(500)^2 \times 0{\cdot}05}{0{\cdot}5 \times 150} = 100$ m. with $f = 150$ mm. in a 13×18 cm. outfit. Accordingly the error in elevations,

$$H = \frac{D}{f} \cdot \Delta y + \frac{y}{f}\, \Delta D = \frac{500}{150}\, 0{\cdot}05 + \frac{50}{150}\, 0{\cdot}5 = 0{\cdot}33 \text{ m.},$$

which involves a maximum y ordinate of 5 cm.]

4. In what connection is the stereoscope used in surveying? Explain how it may be used to obtain contour lines and distances from a pair of photographs taken from adjacent points. (I.C.E.)

ARTICLE 4: AERIAL SURVEYING

Fundamental principles. Aerial surveying may be said to be the evolution from the *ideal case* of vertical photography to the *general case* of resection in space by oblique photography.

The ideal case of producing a map of a plane area is easily conceived as a mosaic composed of strips, each containing a series of exposures, the scale being the ratio of the focal length f of the camera to the constant height H of the aircraft.

The following ideal conditions would be involved in this simple problem in *planimetry*, as distinct from topography, which in addition takes into account the configuration of the surface of the area surveyed: (*a*) Straight courses in specified directions, (*b*) over one or more specified ground positions, (*c*) at a constant height, and (*d*) without tilt of the machine, fore, aft, or laterally, the strips corresponding to precisely parallel courses with longitudinal overlap to allow the simplest stereoscopic examination so essential to the identification of points.

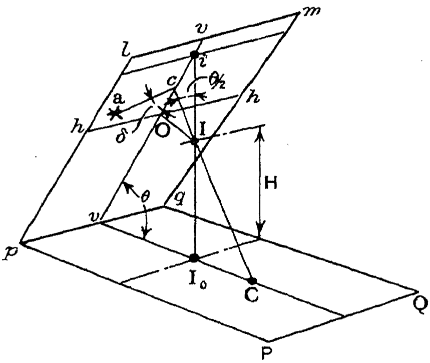
In practice, however, gravitational devices are affected by the accelerations of the machine and thus do not indicate the vertical unless the aircraft is flying a straight course at uniform speed; variation in flying height varies the scale of the map, and lateral tilt introduces unphotographed gaps between adjacent strips, while longitudinal tilt causes gaps between the several photographs of the strips. Also the action of the wind upon an aeroplane flying a straight course may cause the machine to be inclined to the direction of flight, and thus " drift " may deform the photographs of a strip from *file* to *echelon*. Even if wind is correctly considered in calculating a course, the nose of the aircraft will not point in the direction on account of the relative velocities of the wind and the aircraft, the effect being known as " crabbing ". Devices are employed to reduce the first and last of these navigational effects, while systematic procedure will largely eliminate the gaps.

It is, therefore, evident that ordinary mosaics, though serving many useful purposes in reconnaissance for engineering, agricultural and geological projects, do not represent the areas on a definite scale apart from the errors due to tilt, etc.; but, on the other hand, the art of the mosaic

has been so completely developed or controlled, that by rectification some excellent maps have been produced.

Perspective. Opportunely the case of oblique photographs may be considered as a means of introducing a few definitions.

In Fig. 112 the principal axis IO of a camera is shown perpendicular to a photographic plate, which, at exposure, is inclined at an angle θ to the horizontal ground plane $PQpq$, I being the camera position and O the principal point. The points iI_0, vertically above and below I are known as the **plate** (or **photo**) **plumb points** and the **ground plumb points** respectively, the angle OIi being also θ. If this angle be bisected by a line through I, the bisector will determine the plate and ground **isocentres**, c and C respectively; and it can be shown that angles measured on



. 112.

inclined photographs from the plate isocentre will be equal to the corresponding angles measured on the ground from the ground isocentre. Wherefore any distortion of the pictured areas due to tilt must be radial from the isocentre. On the other hand, height distortions will be radial from the plumb points; and in consequence it is necessary to know the effect of assuming that tilt distortions are also radial from the plumb point. By geometry it can be shown that the error on this assumption is very nearly $ca(\frac{1}{2}\theta)^2 \sin 2\delta$, which has a maximum value of $ca(\frac{1}{2}\theta)^2$ when $\delta = 45°$, δ being the angle between the isocentre c and the pictured point a.

Now it is obvious that the principal point O, being defined on the photographs, would be the most practical origin or centre, and it is convenient to use this provided the tilt is small and the ground height variations limited. It can be shown with reference to terrain points of height h that the error in assuming height distortions are radial from O is $f \,.\, h/H \tan\theta$, while for the pictured point a the tilt distortion is $ca(\frac{1}{2}\theta)^2$, as before.

When the principal axis is vertical, as in mapping from "verticals", the points O, c, and i are coincident, being the common origin of the radial assumption. Also it follows that the scale at the level of h is

$$\frac{f}{H-h}, \ \theta \text{ being } 0.$$

The basic difficulty in aerial surveying arises from the fact that central projection introduces distortions due to the height h of an object, and this will at once be evident on assuming a horizontal negative with respect to a lens of focal length f at a height H above datum, rays being drawn between the negative and the elevated and datum positions of a point A, appearing as at A_1 in the datum plane, and pictured as a and $\mathbf{a}_1$ accordingly.

Here the following simple relations will exist between the pictured and actual distortions aa_1 and AA_1:

$$\mathbf{aa}_1 = AA_1 \frac{f}{H}; \quad \mathbf{aa}_1 = o\mathbf{a}_1 \frac{h}{H}.$$

Also the distortion of a point projected on to the datum plane will be $h \cot \frac{1}{2}\theta$, where θ is the angle subtended at the lens; and this suggests that the distortion could reach a limit of the scale value of h at the margins of the pictures with an ultra-wide angle lens.

Thus on a scale of 1 : 25,000, the distortion on the pictures for a height of 500 ft. could be 0·24 in., or 2·40 in. on a scale of 1 : 2500.

Apart from the above, the actual variations from the ideal conditions premised in vertical air photography are (1) uncertainty of height of flight H; (2) tilt of aircraft or inclination of photographs; and (3) inclination of air base.

The practical aspects of these will be considered in a later article, where the geodetic elements essential to mapping will be treated under "Ground Control".

The general problem may therefore be expressed as follows: Given three terrain stations of known position and elevation, or points previously determined on photographic plate pairs, to determine from the corresponding pictured points the co-ordinates in space of the central point and the orientation of the photographs.

The problem thus becomes analogous to plane tabling by "radiation" from the central point, which may be the principal point, the isocentre or the plumb point; and in the general case, points that appear in at least two adjacent photographs may be determined by the intersection of rays from the central points, as in plane tabling by "intersections". Correlation with the terrain must be through the medium of ground control in order that the scale and corrections may be applied.

Radial method. Let it be assumed that the photographs have been taken with a small tilt so that no appreciable error intrudes with the assumption that lines on the photographs and the ground are radial from the respective principal points O_1, O_2, etc., or more conveniently 1, 2, 3, etc. Thus Fig. 113 is a short strip photograph containing *five* overlapping photographs with a, b, c, d, e, and f, clearly identifiable points selected as *minor* control points, each in the common overlap of three photographs. Near the ends of the strip are ground control points P and Q, the co-ordinates of which are known, and hence the true length and bearing of PQ.

Now on the photographs only approximate distances are known, while, owing to variations in ground heights, the scale is variable, with an approximate value available, presumably at the mean ground-level.

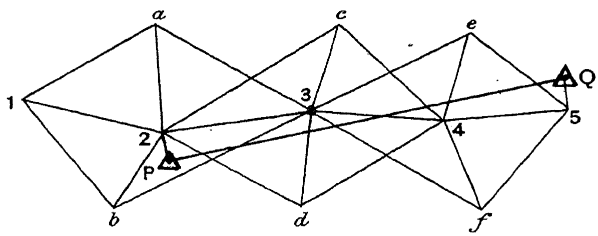


FIG. 113.

Consequently analysis must be made first with an unknown scale. If $P2$ is measured and the approximate height of the aircraft is known, the approximate length of $P2$ may be found, given the focal length f. If the angles between the principal point bases to all minor and ground control points have been measured, say with the **radial triangulator**, the chain of triangles may be solved with the Sine Rule, introducing adjustment of triangles by the Log. Sine process if this is warranted (see p. 422). A traverse may then be plotted along the principal points ; the angles are known, and $P2$, 23, 34, 45 and $5Q$ are known relatively, but to an unknown scale. Also the differences in latitude and departure of P and Q can be found from the photographs, but likewise to an unknown scale. But since PQ is known, a scale ratio may be found, and with this ratio the co-ordinates of the principal points and minor control points can be calculated. For elevations, a number of vertical points must be found.

The well-known Arundel method is a graphical application of the above radial principle.

Ground elevations. Now the fact that variations above datum of the earth's surface varies the scale from f/H to $f/(H-h)$ introduces a very important principle, which is the basis of determining elevations by stereoscopic measurement.

Reconstruction from photographs follows from observation of plate pairs, or the superposition of a pair of pictures in complementary colours, blue and red, as in the anaglyphs viewed by correspondingly coloured spectacles, or used in the Zeiss aeroprojector. As already stated, stereoscopic interpretation is a great asset in the identification of pictured points, and is used even in planimetry.

In Fig. 114, I_1 and I_2 are two adjacent camera positions from which photographs have been taken with the principal axes vertical; the stereoscopic base, a distance B between those stations, being B at a height H above datum. The horizontal traces v_1v_1 and v_2v_2 represent the photographs, the principal points of which are O_1 and O_2 respectively. If rays are drawn from the positions I_1 and I_2 through the left- and right-hand

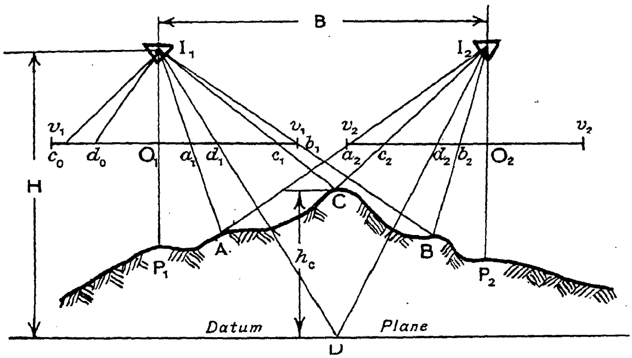


FIG. 114.

extremities of the photographs to meet the ground surface at A and B, they will meet the corresponding rays through the pictured points a_1 and b_2.

The extent of stereoscopic relief will be represented on the ground by AB. Assuming that all points concerned are in the vertical plane of the figure, consider a point C at a height h_c above the corresponding point D in the datum. If lines be drawn from I_1 parallel to I_2c_2 and I_2d_2 to meet the plane of the photograph from I_1 at c_0 and d_0, c_1c_0 and d_1d_0 will measure the separation of the pairs of images and will give the absolute parallaxes of C and D, p_c and p_d respectively. Absolute parallax is unaffected by the actual spacing of the photographs or alteration in the base of the stereoscope, for if the figure is shown with half the spacing, the direction of the ray to any point from I_1 or I_2 is unaltered. Since $I_1O_1=f=I_2O_2$, it follows from the similar triangles that $c_0d_0/f=B/H$, so that $p_d=fB/H$, while similarly, $p_c=fB/(H-h_c)$,

or generally,
$$p=\frac{fB}{H-h}. \qquad (1)$$

H.H.S.

Hence if the air base B and the height of flight H are known, the parallax determined for a point from a pair of photographs will give the measure of a height h above a datum plane. Further, the difference in parallax between two points A and C, namely δp, is $p_A - p_C$.

$$\frac{\delta h}{\delta p} = -\frac{(H-h)^2}{fB}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

but, since the points will be relatively near the datum plane, h being small compared with H,

$$\delta h = \frac{H^2 \, . \, \delta p}{fB}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Further, if μ be the scale of the photography, f/H,

$$h = -\frac{f}{2\,.\,B}\,\delta p. \quad \ldots\ldots\ldots\ldots$$

In all these equations the distances are in a common unit, the metre conveniently, whereas in practice the parallax p will be in mm., f in cm., and h in feet or metres, the scale being universally expressed by the representative fraction.

Air base and overlap. The meaning of overlap in strip photography is a common source of confusion. Let w be the width of the plate and k the ratio of the longitudinal overlap, often 60 per cent, with reference to two vertical camera positions at the extremities of an air base of length B at a height H above datum. In Fig. 115 it is evident that of the length L photographed from the extremities of the base, the length M will be covered by the overlap.

Here it is evident that $$M = \frac{kwH}{f}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$2B + M = L; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

also $$L - B = H\frac{w}{f}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

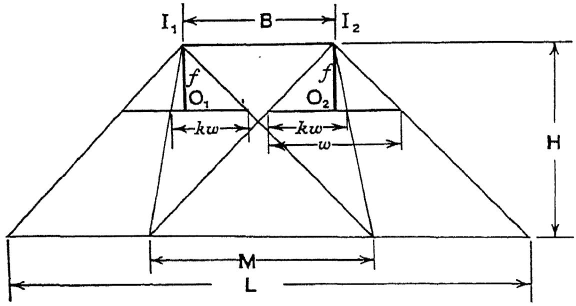


FIG. 115.

and on substituting for M from (1),

$$B=\frac{wH}{f}(1-k) \; ; \quad .. \qquad .(5)$$

or, introducing the scale μ, $\quad B=\frac{w}{\mu}(1-k).$

When an average ground height is assumed, $H-h$ must be written for H.

Example†. The following data refer to an aerial survey :

Camera : Working focus, $8\frac{1}{4}''$, width of plate, $7''$, 60 per cent overlap.

Altimeter : Reading 12,000 ft., height of scale point, 1300 ft. above M.S.L., though a considerable area was at 1800 ft. above M.S.L.

Determine the scale of the network, the length of the air base, and the absolute parallax of a point 1550 ft. above M.S.L. (U.L.)

Assuming an average ground height of 1500 ft., the scale will be

$$\frac{8{\cdot}25}{12(12000-1500)}=\frac{1}{15272};$$

say 1 : 15000 by the relation on p. 242.

Also by writing $H-h$ for H in (5),

$$B=\frac{wH}{f}(1-k)=\frac{7\times 10500}{8{\cdot}25}\times 0{\cdot}4=3564 \text{ ft.}$$

The absolute parallax of the point will be

$$\frac{fB}{H-h}=\frac{8{\cdot}25\times 3564}{12(10500)}=0{\cdot}233''=5{\cdot}918 \text{ mm.}$$

QUESTIONS ON ARTICLE 4

1†. The following particulars refer to a preliminary survey for a highway, made from an aeroplane flying along a course due east.

Camera : Working focus, $7''$; plates, $7''\times 7''$. Longitudinal overlap, 50 per cent ; lateral overlap, 25 per cent. All vertical exposures.

Conditions : Flying speed, 110 m.p.h. Wind velocity, 25 m.p.h., from N.W. Altimeter reading, 9000 ft. Average altitude of ground surveyed 2000 ft. above sea-level.

Calculate (a) the length of the air base and the area covered by each photo, and (b) the time interval between the exposures. (U.L.)

[(a) 3500 ft. ; 0·66 sq. ml. ; (b) $26\frac{1}{4}$ sec.]

2†. Describe a stereocomparator suitable for determining altitudes from pairs of aerial photographs.

Deduce a formula for the altitude of a point in terms of (a) the length of the air base (assumed horizontal), (b) the working focus of the camera, (c) the parallax, and (d) the height of the air base above mean sea-level. (U.L.)

3†. Calculate the error in elevations due to an error in parallax of 0·02 mm. when flying at heights of 6,000, 12,000 and 15,000 ft. for scales of 1 : 10,000, 1 : 20,000, and 1 : 25,000 respectively, the focal length of the camera being $7\frac{1}{4}$ in., the plates also $7\frac{1}{4}$ in. × $7\frac{1}{4}$ in., and the ground inappreciably above mean sea-level. The longitudinal overlap may be taken at 60 per cent.

[82·5 ft.; 165 ft.; 206 ft.]

4. Explain briefly how photographs taken during an aerial flight may be used for surveying purposes, giving one method used to eliminate the effects in variation in camera tilt and camera height.

Give a list of the advantages and disadvantages of this method of surveying as compared with the older method. (I.C.E.)

5. (*a*) Define principal point, plumb point, isocentre and principal line on an aerial photograph.

Prove that the product of the perpendiculars from homologous points on to the horizon traces is a constant.

(*b*) Find an expression for height distortion, and show that tilt distortion is radial from the isocentre.

(*c*) Two points P (2,400 feet above map plane) and Q (1,200 feet below map plane) have co-ordinates $x = +3''{\cdot}06$, $y = +2''{\cdot}93$ and $x = -2''{\cdot}50$, $y = -4''{\cdot}74$ on an untilted aerial photograph taken at a height of 10,000 feet with the principal point as origin.

The principal point is 7″ from the perspective centre.

Find the differences between the true distance and direction of PQ on the ground and the distance and direction of PQ according to the photograph. (U.C.T.)

[534 ft.; 1° 43′.]

ARTICLE 5: AERIAL SURVEYS

The general procedure in the conduct of an aerial survey may be said to consist of the following, which in any project are interdependent and involve close co-operation: (1) ground control; (2) photographic navigation; and (3) cartography.

(1) **Ground control.** The ground survey consists in supplying those geodetic elements by which the photographic data are correlated with the terrain surveyed. It is analogous to the triangulation system of a trigonometrical survey, though it may incorporate points actually selected from the photographs. It also introduces *horizontal* control and *vertical* control, though these are not so easily differentiated as in the case of topographical surveying. The extent of the ground control is determined by (*a*) the

objects and scale of the map, (*b*) the navigational control, and (*c*) the cartographical process by which the maps will be produced.

The extent of ground control ranges from little or none in reconnaissance, up to as many as four points per photograph in some work. Frequently the points will be the minor triangulation stations of a State survey, such as the tertiary stations of the Ordnance Survey. As the scale increases, these will be supplemented by salient points of known elevation, actually or artificially interpolated, while in the largest scales a triangulation net will be established. In vertical photography the points may be as far as five miles apart for a scale of 1 : 25,000 and as close together as four hundred feet for a scale of 1 : 1500. A chain of triangles along the main rivers provided the ground control in the Irrawaddy Delta Survey (1924), the final scale being 3 inches to 1 mile.

The selection of control points or stations should be arranged between the ground and flight personnel ; and, if necessary, photographs of a trial control system should be made with the view of interpolating additional points. Also automatic flying control reduces the number of points, and thus in a large-scale survey, a ground control point near each end of a strip may suffice, while with manual control the number may be reduced to two per strip when the series overlap laterally.

Control points should be readily identifiable on the photographs and should be rendered conspicuous by marking them with whitewash, heaps of stones, or trenches. In this connection, it should be noted that a square of 0·02 inch side on a photograph corresponds with a square of 40 ft. side on a scale of 1 : 25,000 or 4 ft. on a scale of 1 : 2500. Although road and railway crossings, bridges and isolated buildings are easily identified, much skill and practice is demanded in this connection, and pages could be devoted to the difficulties that arise.

When levels or contours are to be determined, the triangulation stations and supplementary points will be augmented with spot heights such as bench marks and barometric levels, the number being between six and twelve per overlap, or nine to eighteen per photograph in simple vertical methods. The number is regulated by the amount of surface character, featureless terrain requiring less.

(2) **Photographic navigation.** Under this heading will be considered (i) mode of flight ; (ii) height of flight ; (iii) tilt of aircraft.

(i) **Mode of flight** is closely related to the method adopted for the survey. For example, the War Office policy is to control the flight and photography rather than to adopt the elaborate plotting machines so largely used by Continental surveyors. When no maps are available, controlled flight for medium-scale maps consists in flying " navigating strips ", enclosing rectangles of about 60 miles by 20 miles, the terrain being divided into squares of about 20 miles side by tie strips. These are used in a preliminary mosaic framework, and the track lines and

permissible deviations for the filling-in strips are marked. The pilot and navigator are supplied with photographs mounted in sets of four for each strip, in order to ensure that there is some ground-point near the track line that can be identified from the air. Many recommend some form of strip flying for most purposes, including small-scale maps and certain engineering plans, the general direction of the centre line being followed in the latter connection. Incidentally, manual flight is best for small areas.

(ii) **Height of flight.** Not only has the surveyor no reference bench mark, as in the case of ground surveying, but even when an automatic pilot is working, the flying height may vary on account of changes in atmospheric conditions.

As already stated, the altimeter has many limitations, and is commonly correct to only about 200 ft. in showing absolute heights. Usually a statoscope, or differential aneroid, is used in addition to show the variations in altitude between exposures. The type ordinarily used shows height variations within a few feet of the starting height and the instrument is synchronised with the photography. Although the absolute height is not known precisely, the variations in height from photograph to photograph facilitate the determination of scale variations by direct measurements on the photographs. The liquid used in the statoscope is amyl alcohol, which has a low freezing point and a specific gravity less than that of water ; and it is claimed that some patterns will record variations to within half a metre. Sometimes the outfit consists of one statoscope in the pilot's cockpit to assist him in his observations, while another is photographed by a recording camera which is synchronised with the survey camera, a clock and counter being also photographed.

Finally, the air base between two adjacent camera positions may be inclined by the effect of air currents or pockets and, in mapping, this inclination must be taken into account.

(iii) **Tilt of aircraft.** It has been established that a specialist pilot can limit the tilt from the vertical to 2° without the aid of gyroscopic stabilisation. The experiments of the R.A.F. with "Three Axes Automatic Control" have shown that stabilised flight can be used advantageously, and it has been stated that the tilt can be reduced to $\frac{1}{4}°$ by an application of this control. Doubtless the use of a so-called "automatic pilot" is a great economy where the ground control is limited, particularly in large-scale mapping, though in the U.S.A. its use is not considered satisfactory. Nevertheless, the course is maintained more nearly level, thus reducing the error due to inclination of the air base. Navigation has also been controlled by wireless, and the camera level stabilised gyroscopically.

Longitudinal and lateral overlaps of 60 and 25 per cent respectively are considered satisfactory for giving stereoscopic examination to vertical photographs. The effect of tilt cannot be detected when it is as small as

2° in vertical exposures, though it is at once apparent in obliques, particularly when these are compared with the corresponding rectified views. It can be determined by instruments such as McLeod's tilt-finder. The consequent loss of surface increases with the scale, and for 2° longitudinally and laterally is about 9 per cent for a scale of 1 : 20,000. A device for recording tilt is operated by a gyroscopically controlled light, the image of which defines the plumb point at the instant of exposure. The effect of tilt is eliminated by the graphical or mechanical methods of plotting, or in rectification by the epidiascope.

Cartography. The cartographic methods will be determined by (*a*) the scale and objects of the survey, (*b*) nature and development of the terrain, and (*c*) the economic factors which influence the cost of production, and possibly future development under the responsible survey authority.

It may thus appear that the following conclusions should rather serve as a preamble to this article; but that, on the other hand, would suggest specialised treatment of the subject, which is outlined here only in an elementary manner with particular regard to the simpler methods.

In general, however, the choice of methods may be grouped as follows, though numerous exceptions might be cited :

Large-scale mapping (1/500 to 1/5000) : Very large scales plotted semi-graphically from rectified verticals based upon dense ground control. Cadastral maps, such as the 1/2500 Ordnance Survey Revision from rectified verticals, certain stages of plotting being eliminated by instrumental methods.

Reduction of heavy ground control by utilisation of photographic control points by means of the radial triangulator for polar co-ordinates and the stereocomparator for rectangular co-ordinates.

Contoured sheets from verticals and quasi-verticals with the stereoplanigraph.

Medium-scale plotting (1/5000 to 1/30,000) : Graphically from verticals by radial methods, such as the Arundel, using simple methods, controlled navigation, and adequate ground control, the process being limited by considerable variations in ground elevations. Stereo-mechanical methods for contour location with reduced ground control, consistent with accuracy in planimetry, especially in the case of large areas. Aero-projector for medium- and small-scale maps when the stereoplanigraph is unnecessarily precise, or for contours and details under the control of the stereoplanigraph.

Small-scale mapping (1/30,000 to 1/50,000) : Oblique photography, particularly in underdeveloped country where contours are required. Multilens groups or ultra-wide angle verticals, unless larger scales are also required.

Arundel method. This method of mapping from air photographs may be described concisely as follows.

The geometrical basis of the radial method was discussed in Article 4 (p. 240), Fig. 113 of which will be followed in the procedure as nearly as possible.

The work consists of (1) inserting the base lines, (2) selecting minor control points, (3) tracing minor control plot, (4) plotting rectified traverse, and (5) inserting detail.

(1) **Base lines.** (i) Using a straight edge centred on the collimating marks at the edges of the prints, rule across to indicate the principal points 1′, 2′, 3′, etc., with pencil or red ink according to the surface of the photographic paper. Each principal point should appear in three prints, thus avoiding the difficulties arising from short overlap or obscured principal points. (ii) Draw the base line through each principal point of adjacent prints, using either of the following artifices :

(*a*) When there is sufficient detail to identify the principal points accurately, the two lines drawn through adjacent principal points should pass through the same points in the detail, and should be inserted finely in red ink.

(*b*) When there is paucity of detail, the pictures should be set in their correct relationship by stereoscopic inspection, preferably with a topographical stereoscope, and the bases should be inserted.

(2) **Minor control points.** (i) Select two clearly defined minor control points, *a*, *b*, in the common overlap of three pictures, in order that a graphical triangulation can be projected with adequate checks. Prick these and ring them around in colour. (ii) Insert fine radials, preferably in red ink, about half an inch in length, from the principal points to the control points, employing stereoscopic examination if possible, since accuracy at this stage will influence all subsequent plotting. Also draw short rays through the ground control points P and Q, which will appear in the first and final print pairs of the strip. (iii) Select a *scale point*, *a*, say, preferably near one end of the strip in the common overlap, *a* being at the average height along the strip.

Now it is not possible to plot the relative positions of the principal points to any known scale, since the ground heights along the strip vary, and the resulting effects must be eliminated in the process of plotting.

(3) **Minor control plot.** (i) Place a strip of tracing paper over the first print (Ph. 1), and prick through the scale point *a* and the principal point 1′ (*P.P.*1′) ; then insert the base line and a ray through *b*, using blue ink or a hard pencil. (ii) Place the tracing over Ph. 2 with the traced base line coincident with that of the print ; then, moving the tracing along the base line, make the ray through *a* on the print pass through its plotted position on the tracing. (iii) Secure the tracing thus : mark

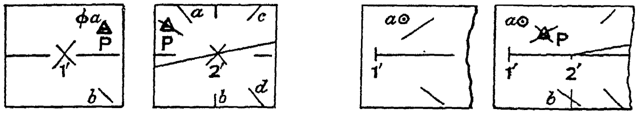


FIG. 116.

*P.P.*2 and the base 2′–3′, and draw rays through *b*, *c*, and *d* (Fig. 116). (iv) Continue the process with Ph. 3, checking by the accuracy with which the third ray cuts the intersection *b* as fixed by Ph. 1 and Ph. 2. Likewise note that when the rays from Ph. 4 are traced, these pass through the previously plotted positions of *c*, *d*, etc. Triangles of error that arise, mainly from occasional tilts of the aircraft, may have to be adjusted, though the difficulty will not be serious if the tilts are small. The minor control plot will show the principal points 1′, 2′, 3′, 4′, 5′, in their relatively correct positions to an unknown scale, and the amount of height distortion of any point may be observed by orienting the tracing over the corresponding print.

(4) **Rectified P.P. traverse.** (i) Plot the ground control stations *P* and *Q* from their known co-ordinates, preferably using graticules of latitude and departure (Fig. 117). Observe that *P* and *Q* will be intersected from two points in making the minor control plot, giving false points *P*′ and *Q*′, which indicates that the scale of the minor control plot is in the ratio of *P*′*Q*′ to *PQ*. (ii) Insert the rectified principal points on the grid as follows: Select any convenient point *O*′ on the minor control plot, and join *P*′ and *Q*′ to *O*′. Place the tracing over *PQ* with *P*′ on *P* and *P*′*Q*′ along *PQ*, and prick through *O*′. Repeat the process but with *Q*′ on *Q* and *Q*′*P*′ along *QP*, thus obtaining *O*. The triangle *PQO* is similar to *P*′*Q*′*O*′. (iii) Finally use the process in plotting the principal points. Place *P*′ on *P* with *P*′*Q*′ along *PQ* and *P*′*O*′ along *PO*. Prick through the points, 1′, 2′, 3′, 4′, etc. Repeat with *Q*′ on *Q*, *O*′ on *O*, and draw rays from *P*, *Q*, and *O*. Observe that each principal point should be fixed by the exact intersection of three rays, and if any ray

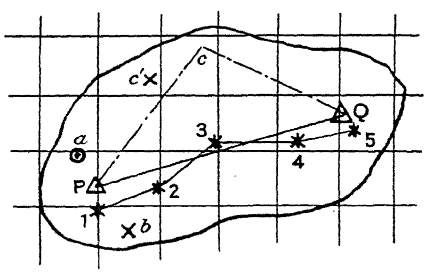


FIG. 117.

gives a weak intersection, some other point R, more convenient than O', should be selected. (iv) Finally plot the principal line traverse in its correct position on the grid: 1, 2, 3, 4, 5. (See Note below.)

(5) **Detail.** (i) Select about ten distributed points from each print which can be identified on each of three photographs, mark these with dots and number them in red. If there is an adjacent strip, points in the lateral overlap should appear in six photographs. (ii) Prepare a tracing of the graticule and the rectified principal point traverse 1, 2, 3, 4, 5. Fix the selected points of detail with three rays on the tracing, indicating their positions appropriately, preferably in blue, and likewise fix points in the overlap between strips from six principal points. This plotting process corresponds to the original minor control plot but on a known scale.

(iii) Insert additional points ($\frac{3}{4}''$ to $1''$ apart), completing the detail by intersections, provided the foregoing points are accurately determined. (iv) Finally trace the details from the photographs, using the plotted detail points as control. If any appreciable error occurs from discrepancy in the assumed approximation of the photographic and cartographic scales, fix a few extra points by intersections. Use stereoscopic interpretation during the mapping. (v) Complete the detail by joining up in black lines, and produce the map by direct printing or by reproduction.

Note. When two ground-control points are not available on each strip, the following procedure must be followed, provided that in two adjacent strips, overlapping laterally, a ground station, preferably a triangulation point M, appears at the beginning of Strip 1 and another N at the end of Strip 2. Select points m and n near the ends of the strip close to the edge remote from M and N respectively, both in the common overlap, these points being subscribed with the numbers of the strip; as, m_1, n_1, m_2, n_2. Select a point S in Strip 2 near the edge opposite to m_1, and in making the minor control plots intersect these points and the triangulation points. Bring the scale of Strip 2 to the unknown scale of Strip 1, in the following manner: Using a piece of tracing paper, large enough to cover both strips, trace off M, m, and n from Strip 1, and, placing the tracing over Strip 2 with m_1 over m_2 and m_1n_1 along m_2n_2, draw rays through S and N from m_1. Repeat the process from n_1, fixing S and N to the scale of Strip 1. Now place the tracing over the graticules on which M and N are plotted to scale, and, by intersections, fix the correct positions of m, n, and S. As before, place M on the tracing over M on the grid with a common direction for MN, and draw rays through m, n, and S. Repeat the process from N. Each minor control plot can now be adjusted independently to the scale of the graticules, and the principal points inserted upon it.

QUESTION ON ARTICLE 5

1†. Describe, with the aid of sketches, a radial method of mapping from vertical air photographs. Discuss the assumptions made, indicating the limitations of the method ; and show in what way developments are widening the scope of its application. (U.L.)

ARTICLE 6 : CARTOGRAPHICAL METHODS

Although the title of this article may suggest treatment of the various plotting instruments and machines which have been introduced, particularly since 1920, a brief discussion of only the simplest forms is possible in a work of this nature, and the student is referred to a treatise on the subject and the pamphlets issued by the various makers.

Stereoscopy played an important part in the latest developments of ground phototopography, but the inception of the aerial method has led to numerous modifications and improvements. Closer investigation has been made into what may be styled stereoscopic perception, the practice of which has become essential to aptitude in the use of the various instruments.

The wandering mark of the earlier stereocomparators has been superseded by other means of observing stereoscopic depth, and the floating mark may consist of a dot, arrows, or a pair of ⊣⊢ or ─┘┌─, both of which should merge into a cross, irregularity in this respect indicating non-correspondence in the marks, either in themselves or the surrounding area. The point is particularly useful in contouring, indicating in the plastic the point otherwise fixed by the foot of a levelling staff. Glass plates covered with grid squares, scribed diagonally, are also used, the relative movement of the grids over the pictures serving as a means of estimating differences of parallax.

A number of variables influence the stereoscopic reconstruction of a plate pair, even in the *normal case* of vertical photographs taken from the ends of a truly horizontal base. The human eye has much latitude, and a plastic view will be more readily conceived than the presence of the less familiar wandering marks. Thus the fusion of the relief and the lines in a plane will suggest perfect correspondence.

Fourcade, in his " correspondence " theory (1926) showed that if *five* points can be accurately identified on a plate pair, the photographs can be set in their correct mutual relationship without reference to ground

control, thus establishing a relation to the air base when this is not horizontal. The five elements of condition are: (*a*), (*b*) swing of each picture about a principal axis containing the principal point of the photograph; (*c*), (*d*) cant of each photograph front and back in the epipolar plane containing the principal point; and (*e*) differential tilt from the rotation of the photograph about the air base. This theory has been applied in the design of various observing and plotting machines.

Stereoscopes and stereometers. Many favour the use of the parallactic grid in less accurate work from verticals. The grid consists of a squared graticule, ruled diagonally on a sheet of optical glass, the directions being styled conveniently **N.E.** and **N.W. lines.** A pair of such grids is arranged with reference to a common base line, which is also used in setting the photographs, the principal point of each and the image of the one in the next being in this base. The grid should appear to be horizontal and in the plane of the photographs, and this will be realised if the line between adjacent crossings of the grid lines is parallel to the eye base. Otherwise the lines of the crossings may fuse stereoscopically, but will appear to be at different levels.

Barr & Stroud's topographical stereoscope is a simple instrument in which the air base is reduced by parallel mirrors. Vertical photographs with 60 per cent overlap are so oriented that fusion can be obtained. Each photograph is covered with a grid, and by adjusting the spacing of these grids so that they appear at ground-level, first at one point *a* and then at another point *b*, the difference of level between *a* and *b* can be estimated. The chief use of the instrument is in inserting contours from a number of points of known elevation in the photographs. The parallax scale, which shows the movement of the grids, is graduated in mm., but estimation is possible to tenths of a millimetre.

The **Zeiss folding mirror stereoscope** is similar in type, and is likewise used in the preliminary examination of air photographs. This stereoscope is fitted with a stereometer which may be used for parallax measurement from a pair of photographs secured with adhesive tape, preferably transparent. A parallel motion is sometimes fitted in order that the stereometer may be moved to maintain a direction parallel to the base line of the pair of photographs. Either of two floating marks, one a dot, may be used according to convenience, and a micrometer is fitted at one end in order to vary the spacing of the marks, the fused mark appearing at ground-level of a given point when each half-mark covers the image of that point.

The **Precision topographical stereoscope** of Messrs. Barr & Stroud is a combined stereoscope and stereometer for accurate observation, the construction obviating the difficulties arising from an attached stereometer. The photographs are set in this instrument with their principal points at the centres of turntables, these centres being on the base line of

the grids, which are lowered over the photographs. Both grids can be moved together, or that on the right can be moved relatively to that on the left, while the photographs can be rotated to some extent, the grid directions remaining fixed. Grid movements can be read to 0·01 mm. by means of a micrometer motion.

Triangulators and comparators. Minor control plotting may be eliminated by rectangular co-ordinates with such instruments as the precision topographical stereoscope or by the radial triangulator, which with negatives or diapositives determines polar co-ordinates with angles to possibly 30″ by estimation, or 1′ directly. When co-ordinates are used, the process is called radial triangulation in planimetry, and aerial triangulation when the third dimension is introduced. The well-known **Zeiss stereocomparator** can also be used in determining the co-ordinates of air photographs which are almost vertical or otherwise have been rectified. The **Cambridge comparator**, as designed by Capt. Thompson, R.E., is specially constructed for the accurate and speedy determination of aerial triangulation, the process being a combination of rectangular co-ordinates and calculation. A special feature of this instrument is its facility for making measurements on prints which show the image of the reseau engraved in centimetre squares on the register plate of the camera.

Rectifiers. Although it is possible to rectify tilted photographs graphically, it is now customary to reproduce them by projection through a rectifier as though they had been exposed vertically. Some rectifying processes correct for tilt and provide for enlargement or reduction, while others rectify the photographs of multi-lens cameras, the bounding obliques being rectified through angles much greater than those of vertical photographs. The **Zeiss automatic rectifier** comprehends tilt up to 40° and gives enlargement up to four times and reduction to one half, the instrument introducing the " parallelogram ", which allows rectification to be effected rapidly to four ground points. The **Wild rectifier** is essentially the same as the Zeiss model. The **Barr & Stroud epidiascope** is designed to rectify photographs in planimetric mapping, as in the revision of cadastral maps.

Plotting machines. Plotting machines are designed to give reconstruction in space of a stereoscopic pair, which is set so that each is examined in the relative position it occupied at exposure with respect to the air base or the horizontal reference plane. The orientation of the photographs involves, summarily, three stages : (*a*) independent setting, so that each picture is set with the principal point on the principal axis in the position the plate occupied in the focal plane ; (*b*) mutual setting, in which each picture is adjusted to its correct position with the other, giving stereoscopic reconstruction on an unknown scale and in an unknown direction ; and (*c*) absolute setting, in which the stereoscopic model is in its correct orientation to the required scale, being thus

adjusted to register with the pictures of known and plotted control points.

Certain mechanical and optical features are characteristic of these machines :

(1) The so-called **Zeiss parallelogram**, a device embodied in the majority of plotting machines, is a mechanism which produces a motion similar to that of the floating mark on a stereoscopic pair. (2) The **Porro-Koppe principle**, a device to eliminate the distortion of images through lenses, reverses the direction of the refracted rays in the image space and projects them both back into the object space through a lens of the same focal length and distortion as the camera lens. This process is not incorporated in the **Wild autograph**, since the effect is considered negligible with modern lenses, while with ultra-wide lenses a special glass plate of variable thickness is fitted to eliminate distortion. The Wild Model A5 has been used extensively in many parts of the world for producing contoured maps up to scales of 1 : 1000. A simpler model, A6, is made for plotting on topographical scales from approximately vertical photographs, a maximum tilt of 5° being eliminated in the setting. This instrument is said to be accurate enough for plotting to scales of 1 : 5000 or larger.

The well-known **Zeiss stereoplanigraph**, a universal instrument for plotting on all common scales, can be used for mapping from (*a*) ground photographs with a stereoscopic camera ; (*b*) ground plate pairs with or without parallel principal axes ; (*c*) aerial views with the axes in any direction ; (*d*) multicamera views with the axes set to corresponding directions ; (*e*) strips of any specified length ; and (*f*) single views of flat country by graphical rectification.

Zeiss multiplex aeroprojector. This instrument introduces the principle of anaglyphic projection in plotting medium- and small-scale maps when the stereoplanigraph is inordinately precise, and is frequently used when the latter serves as a control instrument. The apparatus consists essentially of a horizontal bar which carries a number of projectors with which the pictures are projected from diapositives alternately in red and blue, so as to form a model of the landscape, the stages in projecting the picture on the screen from the ground surface being through the negative and the diapositive, which is enlarged in accurate work. The projectors are set in the X, Y, and Z directions with respect to the bar, which represents the air base, and each projector is provided with swing, cant, and differential motions, similar to those in mechanical plotters.

First two photographs are set in mutual orientation (to an unknown scale), and succeeding ones by orienting a third photograph to the second, and so on, up to nine projectors, the result being a space model of the pictures, the scale and orientation of which is not known. Absolute orientation of the space model is effected by means of circular tables, known as control stands, which consist of a central dot at the centre of

a circle, the table being adjusted vertically and fitted below with a needle which can be set over a central point plotted on the master plan. A projection table is provided with a vertical motion so that the exact position of a point can be established in space, the centre of the table being centred exactly over a plotting pencil. The centre of the table provides the position for alternative floating marks of controlled illumination : one a spot and the other a black arrow which can be turned to appropriate positions relative to the ground detail. When the floating mark has been brought to the position of a selected point in space, the pencil marks its position in plan while the height is read off on a vertical scale.

Rectification. *Example*†. Describe with reference to a diagram the general principles of rectification of tilted photographs, introducing both the fundamental scale of the map and the ratio of enlargement.

Let Pi be the plane of the negative, PC the plane of the rectified projection, and PI the plane of the rectifying lens I (Fig. 118). On the scale of the rectified photograph I will be mH/μ vertically above PC, where m is the ratio of enlargement (or reduction) and μ the denominator of the representative fraction of the fundamental scale f/H, f being the focal length of the lens of the camera.

The scale will be consistent along plate parallels, that is, along perpendiculars at points such as o, c, and i, being the fundamental scale along the isometric parallel through c, in which case $\alpha = \beta$, and $\psi = \phi$, with $2\phi = \theta$, the angle of tilt.

Fundamentally, the three planes cited above must meet in a line, the trace of which is P, while as a condition of reconstruction the product of the sides SI, IT of the parallelogram $SITP$ must be constant, where SI and IT are drawn parallel to PC and Pi respectively.

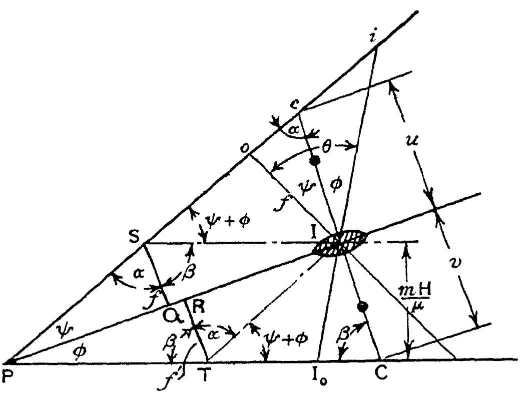


FIG. 118.

Let SQ and RT be drawn perpendicular to PI.

Then $\dfrac{SQ}{u}=\dfrac{PQ}{PI}$, and $\dfrac{RT}{v}=\dfrac{PR}{PI}$, and since $SQ=RT$,

$$SQ\left(\frac{1}{u}+\frac{1}{v}\right)=\frac{PQ+PR}{PI}=1\ ;$$

hence if f' be the focal length of the projector lens, $SQ=RT=f'$.

The relations between the angle of tilt, $\theta=(\psi+\phi)$, and the angles α and β between the axis of the lens and the negative and rectified pictures are evident in Fig. 118.

Three cases may be considered, as follows :

(*a*) **Rectification and enlargement.** Here the angles α and β must be determined, also the obliquity of the line between the pictured and projected principal points :

$$\cos\alpha=\frac{RT}{IT}=\frac{f'\sin(\psi+\phi)}{mH/\mu}=\frac{f'}{mf}\sin(\psi+\phi)\ ;$$

$$\cos\beta=\frac{QS}{SI}=\frac{f'}{f\ \mathrm{cosec}(\psi+\phi)}.$$

Scale of rectified photograph $=m/\mu$.

(*b*) **Simple rectification with lens of focus f.** Here $m=1$, $f'=f$, $=\phi=\frac{1}{2}\theta$, and $\alpha=\beta$, Q and R being coincident.

$$\cos\alpha=\cos\beta=\sin\theta\ ;\ \ \alpha=\beta=90^\circ-\theta\ ;$$

and $u=v=2f$, with c and C the isocentres.

Here $co=CI_0=f\tan\frac{1}{2}\theta$ actually, and a simple mechanism can be arranged by sliding the lens I along the axis IP so that tilts of $\frac{1}{2}\theta$ are given to the negative and the projection by means of a lever system.

(*c*) **Simple rectification with a lens of focal length f′.** Since $m=1$ and $\psi=\phi$, $\cos\alpha=\dfrac{f'}{f}\sin(\psi+\phi)=\cos\beta$.

Example†. Describe concisely a method of revising the 25″ Ordnance maps from aerial photographs, outlining all the essential operations.

(U.L.)

The following description is based upon the work carried out by Messrs. Aerofilms Ltd., and described by Lieut.-Colonel A. Lloyd Brown, R.E., in a paper to the Chartered Surveyors' Institution.

The photographs, taken from a height of 9000 ft. with a 21″ lens, were enlarged to 1 : 2,500 approximately, the plates being $8\frac{1}{2}''\times 6\frac{1}{2}''$.

A number of points in old detail, which could be identified stereoscopically on at least two photographs, were adopted as ground control and in addition a number of minor control points were selected, not necessarily on the old plan. A minor control tracing of each photograph was made, showing the base line and rays through the minor control

points. Incidentally, points known as "slope" control points were selected to reduce height distortions in mapping. Like the other points used, these were selected after stereoscopic examination.

The minor control tracing for the first photograph is shifted about over the master plan, until each ray passes through the corresponding point, and the base and control points are pricked through.

When all the control points have been established, the detail is traced from the photograph within separate triangles as determined by three slope control points. An epidiascope is used in projecting that part of the photograph on to a glass screen placed under the relevant part of the plan, and, by means of the tilt and lateral movements available, the vertices of the triangle are made to coincide with the corresponding points on the plan. The detail is then traced off within this triangle, and the process is repeated until the area under revision has been plotted.

The work is finally revised in the field, and additional details are noted and plotted wherever necessary.

The Thompson Comparator has been introduced in later work, the reseau being used to check distortions.

Messrs. Aerofilms have also undertaken a considerable amount of work on similar scales for municipal authorities, etc., the results being exceedingly satisfactory.

Example†. Describe concisely the process of contouring from vertically exposed photographs by a simple method with suitable ground control, the prescribed scale being 1 : 25,000 and the contour interval 10 metres.

The following method is restricted to areas in which there is good horizontal and vertical control, and where this is not available, some form of aerial triangulation must be used through the medium of an elaborate instrument.

The procedure may be grouped under the following headings: (1) control points; (2) parallax differences; and (3) inserting contours, a maximum tilt of 2° being assumed and 60 per cent overlap.

(1) Select nine points of known elevation from each photograph (or six per overlap) in vertical control, and in addition at least twelve points from which heights can be obtained by stereoscopic measurement and calculation, the number being reduced if the ground is fairly uniform and increased if the surface features are pronounced.

Vertical control will consist of the heights of trigonometrical stations, prominent bench marks, and on the scale prescribed, clinometer heights supplemented by barometer readings, these last being within 1–2 ft. with sufficient accuracy. Stereo-height control points will be selected in salient positions on the overlap in order that adjustments can be made in circuits, these following the general trend of the ground control points.

If the Arundel method is used, the spacing of the principal points may be found, and the average length of the air base B determined, while the average height H of exposure can be found from altimeter readings, inaccuracies in this respect being largely eliminated by the method.

(2) Measure the parallax differences with the topographical stereoscope, with the photographs set in correspondence, each covered with the parallactic grid, which gives a north (N.) line in addition to the N.E. and N.W. lines. Set the photographs first so that the base line corresponds with the lateral movement of the grids, and move the latter together in the direction of the base line, maintaining their spacing; then move the right-hand grid in the same direction, relative to the left, in order that the parallax difference can be measured. Bring one of the lines to ground-level at the point considered, and read the parallax drum, exercising care that the correct setting has been obtained. Normally, for a point there will be a reading at a N.E. line and then another for a N.W. line, while another will occur at the N. line; and if the images are in correspondence this reading should give the mean reading of the three, affording a check on the accuracy of the observation. Tabulate the points (the trial grid readings) and the mean reading, and determine the absolute parallaxes of the points from the ground vertical control points, using $p=\frac{fB}{H-h}$ either through the medium of (*a*) parallax tables, or (*b*) charts between p and h, or p and $\delta p/\delta h$. Having determined the vertical and stereoscopic spot elevations, insert the contours, using alternate photographs each in turn with its left- and right-hand counterparts.

(3) Mark the control points with number and height on spare index prints to avoid misleading marks on the actual pairs. Set the pairs in correspondence in the stereoscope with respect to the base lines, which have been drawn during the plotting of the plan. Examine the overlap in relief with respect to the index print. Follow the usual rules for spacing the contours on uniform, convex, and concave slopes. Lift the grid from the photograph which is to be contoured, and insert the lines in red, covering thus alternate prints. Trace the contours on to the master map.

QUESTIONS ON ARTICLE 6

1†. Discuss the factors which influence the accuracy of determining elevations from vertical photographs, confining your notes to the following main considerations:

(1) Photographic defects; (2) navigational difficulties; (3) altimeter and statoscope readings, and (4) measurement of parallax.

2†. Describe the Zeiss Aeroprojector, and explain briefly its use in small-scale mapping from aerial photographs.

3†. Describe the epidiascope with reference to rectification of photographs in the revision of cadastral maps.

SECTION IV

FIELD ASTRONOMY

INTRODUCTION

It is convenient to assume that all heavenly bodies are situated on what is known as the **celestial sphere**, since the relative directions, not the actual distances of such bodies, are considered in geodetical astronomy. The radius of this sphere is infinite ; its centre coincides with the centre of the earth (Fig. 119).

The **vertical**, or direction determined by gravity at any point on the earth's surface, when produced, pierces the celestial sphere *above* in a point known as the **zenith**, and *below* in a point known as the **nadir**. The zenith Z, therefore, may be defined as the point on the celestial sphere directly above the observer, and the nadir Z' the point on this sphere directly beneath the observer.

A plane through the centre of the earth perpendicular to the vertical intersects the celestial sphere in a great circle known as the **horizon.** This circle, HH', is styled the **true horizon,** in order to distinguish it from the **sea horizon,** the plane of which is parallel to the true horizon. Hence the *horizon* is a plane tangential to the earth's surface (as determined by the mean sea-level) at the point of observation ; and the *true horizon* is a plane parallel to this horizon, but through the centre of the earth. The horizon, as thus defined, is sometimes styled the *sensible* horizon in distinction from what is known as the *visible* horizon, which latter is a plane parallel to the horizon through the circle of contact of the earth and a vertical cone, the apex of which is the eye of the observer.

The axis of the celestial sphere is the earth's axis produced, and the **celestial poles** are the points in which this axis penetrates the celestial sphere at P and P'.

The **celestial equator** EE' is the great circle of the celestial sphere as determined by the intersection of a plane through the terrestrial equator and the celestial sphere. Its plane, therefore, is midway between the celestial poles and is perpendicular to the axis.

The **celestial meridian** of an observer at any point on the earth's surface is the great circle that passes through the celestial poles and the zenith of the observer. Usually this is represented as a circle in the plane of the paper, $PZH'P'Z'H$ as in Fig. 119.

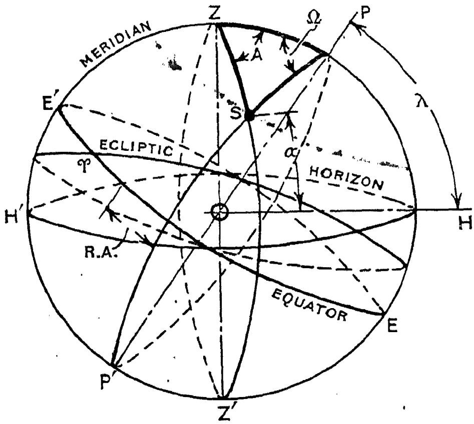


FIG. 119.

The **terrestrial meridian** of an observer at any point on the earth's surface is the great circle passing through the terrestrial poles and the point of observation.

The **prime vertical** is a great circle of the celestial sphere through the observer's zenith and with its plane perpendicular to the meridian. It intersects the horizon in the *east* and *west* points ; and is represented by the straight line ZZ' in Fig. 119.

The **ecliptic** is the great circle of the celestial sphere which the sun appears to describe during the year. It is inclined to the equator at an angle of nearly 23° 27′. The points of intersection of the ecliptic with the equator are the **equinoxes**, and the points on the equator midway between the equinoxes are the **solstices**. The equinoxes, *vernal* and *autumnal*, occur respectively on March 21 and September 21. The solstices, *summer* and *winter*, occur respectively on June 21 and December 21 ; and on those dates the sun's declination is temporarily constant.

Systems of co-ordinates. Two spherical co-ordinates may be used to designate the position of any point on the celestial sphere. The circles of reference consist of a great circle known as the *primary* and a system of great circles perpendicular to the primary, styled *secondaries*. Two systems are commonly employed ; viz., (1) the horizon system, and (2) the equator system, which are also shown on Fig. 119.

(1) **The horizon system.** Here the circles of reference are the horizon and great circles perpendicular thereto, the latter being styled vertical circles. The co-ordinates are known as **altitude** and **azimuth**.

The altitude of a celestial body is its angular distance above the horizon, as measured on a vertical circle. The *zenith distance* is the complement of the altitude α, and is written as $ZS = 90° - \alpha$.

The azimuth is the spherical angle at the zenith between the meridian and the great circle of altitude through an observed celestial body. Azimuth A is measured on the horizon, sometimes from 0° to 360° from *either* the north or the south point, and sometimes, as a bearing, from the nearer of these points.

(2) **The equator system.** Here the circles of reference are the celestial equator and great circles thereto, styled **hour circles** (and sometimes **declination circles**). The co-ordinates are known as **declination** and **right ascension.**

Declination. The declination of a celestial body is its angular distance measured on a meridian *north* or *south* of the celestial equator. In consequence, it is customary to prefix declinations with the letters N and S respectively. Such convention is absolutely necessary, since the letter determines the sign of the declination: *plus* (+) *declinations being those prefixed with the same letter as, and minus* (−) *declinations the opposite letter to, that denoting the hemisphere of observation, the northern or southern, as the case may be.* Writers, in avoiding this statement, limit their matter to observations in *one* hemisphere, and, in consequence, leave access to algebraical confusion. The **co-declination**, $90° - \delta$, is known as the **polar distance**, PS.

Right ascension. The Right Ascension of a celestial body is the arc of the equator between the *vernal* equinox and the hour circle through the celestial body. The meridian through "The First Point of Aries" (♈) is selected in the heavens as the celestial meridian from which are measured Right Ascensions of celestial bodies, as the meridian of Greenwich is arbitrarily selected as the zero from which terrestrial longitudes are measured. Right Ascensions are measured *eastwards* from 0° to 360° (in the direction opposite to that in which the stars revolve around the pole), and are expressed in hours, minutes, and seconds, 24 hours being equal to 360°. The Right Ascension of a star *plus* its hour angle is the **local sidereal time** at the instant.

The declination (δ) and Right Ascension (R.A.) of any celestial body being known, the position of that body in the heavens is at once determinable, the co-ordinates involved corresponding to terrestrial latitude and longitude.

The position of a celestial body may also be determined by its declination and **hour angle.**

Hour angle. The hour angle is the arc of the equator between the meridian and the hour circle of the body under observation. It is the spherical angle at the pole between the observer's meridian and the meridian of the observed body. The hour angle is reckoned *westward* from 0° to 360°, or from 0 h. to 24 h. (being measured on the side remote from the pole in the direction in which the stars revolve about the pole). Expressed in time, it is, therefore, the interval between the instant of

observation of a celestial body and the culmination of that body on the observer's meridian. Local sidereal time at any place and instant is the hour angle of ♈.

The co-ordinates of the observer are his Latitude and Longitude.

Latitude. The celestial latitude is the angular distance of the observer's zenith, north and south of the celestial equator. The *celestial* latitude is equal to the *terrestrial* latitude λ, which latter is the angular distance on a meridian of any point on the earth's surface north or south of the equator.

The latitude of the observer is equal to the *altitude* of the pole. The co-latitude $PZ = 90° - \lambda$.

Longitude. The longitude is the arc of the equator between the meridian of Greenwich (or other primary meridian) and the meridian of the observer. It is the spherical angle at the pole between the meridian of Greenwich and the meridian of observation. Longitude may be expressed in degrees, etc., of arc, or in hours, minutes and seconds, on the equator or on any parallel of latitude.

The spherical triangle SPZ in Fig. 119 is frequently styled the " Astronomical Triangle ", S being the observed celestial body, P the pole and Z the zenith.

Abbreviations and Symbols

The following are used as consistently as possible throughout this book.

L.M.T. - - -	Local Mean Time.
G.M.T. - - -	Greenwich Mean Time.
L.M.N. (M.) - -	Local Mean Noon (Midnight).
G.M.N. (M.) - -	Greenwich Mean Noon (Midnight).
L.A.T. - - -	Local Apparent Time.
G.A.T. - - -	Greenwich Apparent Time.
S.M.T. - - -	Standard Mean Time.
L.S.T. - - -	Local Sidereal Time.
G.S.T. - - -	Greenwich Sidereal Time.
R.A. - - -	Right Ascension.
δ - - - -	Declination.
$\Omega, (\omega)/t$ - - -	Hour Angle, Arc/Time.
α - - - -	Altitude.
A. - - - -	Azimuth.
λ - - - -	Latitude.
L. - - - -	Longitude.

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" The Nautical Almanac " (N.A.) for the Meridian of Greenwich is published annually by H.M. Stationery Office, now in a Standard and an Abridged Edition, the latter being the more convenient to the surveyor. Radical changes have been made in the compilation of the almanac since 1931, and

it is essential that the surveyor should study the explanation appended to the tables, particularly since many text-books adhere to terms and phrases that need correlating with the new arrangement. The effects of the changes in the N.A. are explained with reference to Observations for Time, pp. 279–283.

"The American Ephemeris", a publication of the Bureau of Equipment, U.S. Navy Department, applies in particular to operations in that country.

"Whitaker's Almanack" serves as an approximate substitute for the "Nautical Almanac". The bulk of its contents, however, is irrelevant.

"Chambers' (Seven Figure) Mathematical Tables" contain, in addition to the matter requisite for computations, a number of tables strictly applicable to astronomical observations.

"Hints to Travellers", a publication of the Royal Geographical Society, is indispensable to the pioneer surveyor abroad. It is mainly devoted to field astronomy and route surveying, but contains matter highly important to explorers.

Pocket Books. Molesworth's, Trautwine's, American Civil Engineers', and Kempe's pocket books contain data invaluable to the surveyor abroad. Saegmuller's "Pocket Solar Ephemeris" is extremely convenient in operations with an instrument seldom assessed at its true value by British surveyors, the solar attachment.

SPHERICAL TRIGONOMETRY

A spherical triangle is the figure formed upon the sphere by the intersection of three great circles, the planes of which intersect at the centre of the sphere, intercepting what Euclid defined as a solid angle. The angles are measured between the planes containing the sides, and if tangents be assumed at an angle, these will contain the angle between the planes. As in plane trigonometry, capital letters denote the angles, A, B, C, thus indicated at the vertices, while the small letters a, b, c, denote the opposite sides, which are expressed by the angles these subtend at the centre of the sphere, the corresponding lengths being the product of the radius R of the sphere and the angular value of the sides in radians. Also s denotes the semi-sum of the sides, which is likewise an angle, usually expressed in degrees.

Right-angled spherical triangles. Since it is usual to memorise certain formulae for oblique-angled triangles, the rules for right-angled triangles may be reduced by putting $B=90°$ in the sine and cosine formulae, given hereafter. Otherwise the following artifice may be used.

A convenient mnemonic for writing down ten formulae for right-angled spherical triangles is known as **Napier's five circular parts**:

Sine of any one part, termed the *middle* part $=\begin{cases}\text{product of tangents of the two adjacent parts} & (a)\\ \text{product ,, cosines ,, ,, opposite ,,} & (b)\end{cases}$

Here the angle B is assumed to be 90°, and the five remaining parts of the triangle are shown conventionally in the five sectors of a circle, the thick radius in Fig. 120 indicating the right angle. The sides a and c,

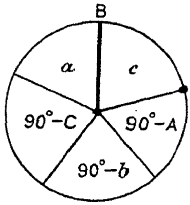


FIG. 120.

actually enclosing the right angle, are inserted in the adjacent sectors, while in the sector opposite to the omitted right angle B, the complement $90° - b$ of the opposite side is shown, the remaining two sectors showing the complements of the angles actually between the sides designated in the sectors on either side; namely, $90° - A$ between c and $90° - b$, and $90° - C$ between a and $90° - b$. Thus

$$\sin a = \tan(90° - C) \tan c = \cot C \tan c ;$$

and $$\sin (90° - b) = \cos b = \cos a \, . \cos c,$$

the former representing the well-known approximation to the "convergence" equation (p. 381).

Oblique-angled triangles. The formulae more commonly used in geodetic problems are as follows :

(1) $\text{Cos } a = \cos b \, . \cos c + \sin b \, . \sin c \, . \cos A.$

(2) $\text{Cos } A = \sin B \, . \sin C \, . \cos a - \cos B \, . \cos C.$

(3) $\dfrac{\text{Sin } A}{\text{Sin } a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}.$

(4) $\text{Tan } \tfrac{1}{2}A = \sqrt{\dfrac{\sin (s-b) \sin (s-c)}{\sin s \, . \sin (s-a)}}.$

(5) $\text{Tan } \tfrac{1}{2}(A+B) = \dfrac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cdot \cot \tfrac{1}{2}C.$

$\text{Tan } \tfrac{1}{2}(A-B) = \dfrac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cdot \cot \tfrac{1}{2}C.$

(6) $\text{Tan } \tfrac{1}{2}(a+b) = \dfrac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \cdot \tan \tfrac{1}{2}c.$

$\text{Tan } \tfrac{1}{2}(a-b) = \dfrac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \cdot \tan \tfrac{1}{2}c.$

The similarity between (1) and (2), and (5) and (6) is obvious, angles merely replacing the sides and introducing respectively a change in the sign and inversion of the last term. Formulae (4) and (5) have their counterparts in plane trigonometry, where lengths replace the sines of the spherical sides. Sin $\frac{1}{2}A$ and cos $\frac{1}{2}A$ might follow (4), but neither is so widely or conveniently applicable as the tangent rule.

Spherical excess. Spherical excess, ϵ, is the amount by which the sum of the angles $(A+B+C)$ exceeds 180° in a spherical triangle of area S, the radius R of the sphere being expressed in the corresponding linear unit :

$$\epsilon = S \times \frac{180^\circ}{R^2} \text{ degrees.}$$

Usually the excess is expressed in seconds, as follows, with the area respectively in sq. ft. and sq. mls., the value of R being taken as 20,889,000 ft.

$$\log \epsilon'' = \log S' - 9{\cdot}3254098.$$

$$\log \epsilon'' = \log S \; + 2{\cdot}1198580.$$

For very large triangles the local mean radius R_m must be used, altering the value of the constant logarithm accordingly, thus :

$$R_m = \frac{R_e(1-e^2)}{1-e^2 \,.\, \sin^2\lambda},$$

Clarke's values for the equatorial radius R_e and the compression e being respectively,

$$20{,}926{,}202 \text{ ft.} \quad \text{and} \quad \frac{1}{293{\cdot}43} \text{ (see p. 377).}$$

ARTICLE 1 : ASTRONOMICAL OBSERVATIONS

The following items will be treated in this article : (I) observing altitudes ; (II) identification of stars ; and (III) computations.

(I) **Observing altitudes.** Altitudes are qualified in accordance with the extent to which they are corrected, being first *observed*, next *apparent*, when instrumental (and semi-diameter) corrections have been made, and, finally, *true*, when the net observational correction for refraction (and parallax) has been applied.

(1) **Instrumental correction.** Apparent index error of the vertical circle of the theodolite is primarily eliminated by means of the clipping screws (p. 37), but the contingency of vertical collimation error, the *actual* index error, can be eliminated by observing with the telescope both normal and inverted (F.L. and F.R.). When Face Left and Face Right observations of a celestial body are impracticable, the so-called index error can be determined after a single face observation has been made. Thus let a vertical angle of 32° 42′ 20″ be observed with the telescope normal (F.L.). Now sight on some well-defined object, and read the vernier.

Let this reading be 20° 16′ 40″. Transit the telescope (F.R.), and again sight the same object. Let the vernier now read 20° 15′ 20″. Then

$$\text{Index error of vertical circle} = -\tfrac{1}{2} \times 1' \, 20'' = -40'',$$

and $$\text{Altitude of celestial body} = 32° \, 42' \, 20'' - 40'' = 32° \, 41' \, 40''.$$

When double-face observations of the sun are taken, the *upper* and *left-hand* limbs are sighted tangentially to the cross-hairs in the N.W. quadrant, thus (|–), and the *lower* and *right-hand* limbs in the S.E. quadrant, thus (– ◡), or vice versa, the sun's image being seen inverted against erect cross-hairs. As is often the case with the diagonal eyepiece, the sun's image will be both inverted and reversed.

If the sextant is used, the sun's image as seen by reflection from the index glass is brought into double-edge contact with its image as viewed through the horizon glass reflected from the artificial horizon; and the altitude is one-half the mean reading of the limb, after allowance has been made for index error (p. 80).

When the altitude level is attached to the clipping arm, the bubble readings should be taken in order to correct for minor displacements that occur during observations, particularly if these involve face reversals or are protracted as in "reducing to meridian". If O denotes a bubble reading at the objective end of the telescope and E a reading at the eyepiece end, then the correction to the altitude will be, algebraically:

$$+\frac{\Sigma O - \Sigma E}{n} d'',$$

where n is the number of ends read and d'' the angular value of a bubble division. In night observations the cross-hairs are illuminated, (*a*) electrically in modern theodolites, (*b*) by the axis lamp and mirror in older complete patterns, and (*c*) by electric torch and improvised reflector across the objective in instruments without accessories for the purpose.

(2) **Refraction.** Refraction is the effect on rays of light in their passage at an angle between media of different densities. The rays passing to the observer through a medium becoming progressively denser are curved concave downwards; and they are curved concave upwards in passing through a medium becoming progressively rarer. The effect, therefore, is to make the altitudes of observed celestial bodies appear greater than they actually are; and, in consequence, the correction for refraction is *always* to be deducted from observed altitudes.

The amount is the same for all bodies, irrespective of distance. It varies from zero for observations in the zenith to about 34′ for observations on the horizon, its amount being approximately 57″ cot α, where α is the approximate altitude. The value of the refraction correction depends also upon the temperature and the barometric pressure, corrections for which under extreme conditions are applied in precise observations, often

in accordance with Bessel's formulae as given in Chambers' Tables. Strictly, refraction, as defined above, is *refraction in altitude* as distinct from *refraction in declination*, which latter is usually calculated in accordance with Chauvenet's formulae : $57'' \cot (\delta + N)$, where $\tan N = \cot \lambda \cos \Omega$, δ, λ, and Ω being respectively the declination, latitude, and hour angle, commonly taken to the nearest degree.

(3) **Semi-diameter.** The centre of the celestial body should always be observed. Stars, however, are points of light, and their centres are readily sighted ; but the sun and moon are large and, in consequence, it is usual to observe their upper or their lower edges, or limbs. The semi-diameter of the sun or the moon is one-half of the angle subtended by the diameter of their visible discs. This angle, however, varies according to their respective distances from the earth, and is, therefore, different at different times of the year. Its value, in the case of the sun, is about $\pm 16'$, the minus sign applying to sights on the upper limb, which appears at the bottom of an inverting telescope.

When face reversals are used as in paragraph (1), *the semi-diameter correction is obviated and the actual index error eliminated.*

The semi-diameter of the sun and the moon for any date appears on page II of the month in the " Abridged Nautical Almanac " (N.A.(*a*)) and the magnitudes of both are shown in separate tables in the N.A. for Greenwich Mean Midnight (G.M.M.), while that of the sun at Greenwich Apparent Noon (or transit) is given both in arc and sidereal time. Chambers gives these corrections in terms of contractions.

(4) Parallax is the difference of altitude of a celestial body, as would result from two simultaneous observations, one at a point on the earth's surface and the other at its centre. It is the correction to be *invariably* added to observed altitudes, in order to reduce such to their corresponding values at a point of observation at the earth's centre. Only the sun, moon, and planets have parallax. Fixed stars, being for all practical purposes infinitely distant, have no appreciable parallax, except a few instances, where the parallax is measured by assuming as a base not the earth's radius, but the diameter of the earth's orbit about the sun.

The **parallax in altitude** with regard to the celestial body S in Fig. 121 is shown by the angle SOr. It is evident in that figure that this angle is zero when the body is in the zenith and a maximum when the body is on the horizon, when it is styled **horizontal parallax.**

The sun's horizontal parallax for the earth's equatorial radius is given for four days of each month in a separate table in the N.A., where that of the moon appears for every half-day of the year. In the N.A.(*a*) the latter is shown likewise, also as the angle subtended at the moon's centre by the earth's equatorial radius. When great precision is required, the *equatorial parallax* must again be reduced by a correction for latitude.

The horizontal parallax of the sun and planets are tabulated for three days

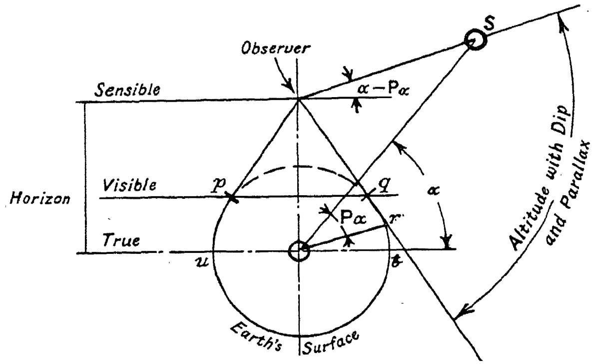


FIG. 121.

on the third page of each month in "Whitaker's Almanack". Approximate values of the sun's parallax in altitude are tabulated in Chambers' Tables, the variations being sensibly constant.

Such values are often computed by means of the relation $P_a = P_h \cos \alpha$, where α is the apparent altitude, and P_a and P_h respectively the parallax in altitude and the horizontal parallax, which latter (though slightly variable) is taken as 8·7″ (or 8·8″), this being the ratio of the earth's radius to the sun's distance.

(The foregoing relation is readily derived by drawing the sensible horizon in Fig. 121 at the earth's surface instead of the observer's eye, thus precluding the exaggeration made in illustrating the various terms and certain distinctions.)

The parallax in altitude P_a of the moon at an apparent altitude α may be computed from its horizontal parallax P_h by means of the formula :

$$\sin P_a = \frac{\cos \alpha \sin P_h}{R} \quad \text{(see also Chambers' Tables).}$$

(II) A **constellation** is a group of stars which is supposed to resemble the terrestrial object after which it is designated and delineated by dotted boundary lines on maps and globes.

Thus, *Ursa Minor*, a polar constellation, is represented by a *little* bear. The individual stars of the constellation are indicated by the small letters of the Greek alphabet primarily, the alphabetical order denoting their relative brightness, or magnitude. Thus, for example, the brightest two stars in the constellation *Orion* are styled α *Orionis* and β *Orionis*, and are also popularly known as *Betelgeuse* and *Rigel.*

Circumpolar stars. A circumpolar star is one the polar distance of which is less than the latitude of observation, and which in consequence never meets the horizon, thus neither rising nor setting, but appearing to revolve about the pole.

A circumpolar star is said to be at *Upper Culmination* (or transit) (U.C.) when it reaches its highest point, and at *Lower Culmination* (L.C.) when it reaches its lowest point. In either of these positions it is in the *true* meridian.

When a circumpolar star reaches the most easterly point of its orbit, it is at its *Eastern Elongation* (E.E.), and at the most westerly point it is at its *Western Elongation* (W.E.).

Star charts. The surveyor should be able to identify stars before undertaking stellar observations. This may be done with the aid of a star chart, or planisphere, many of which are in circulation. A thorough knowledge of the constellations is by no means essential to the surveyor, but he should be able at once to identify the principal stars commonly used in the field.

Identification. The following guide was adapted for the author's "The Field Manual" from "Kempe's Engineers' Year Book":

(i) *Polaris*, or the *Pole Star* (α *Ursae Minoris*), a star of the second magnitude, should be identified primarily in the northern hemisphere. It is the end star of the tail of the *Little Bear*, or as the Americans locate it, the end star of the handle of the *Little Dipper*.

Contrary to the popular idea, *Polaris* is not exactly at the celestial pole, but, like all the stars in the hemisphere, it appears to revolve around it. If it were exactly coincident with the pole, its direction would at once determine the meridian; its altitude, the latitude of observation.

The radius of the circle in which *Polaris* appears to revolve changes from year to year. This radius, the *polar distance*, was 1° 16′ 42″ in 1890; 1° 10′ 27″ in 1910; and 1° 01′ 16″ in 1940. It will continue to decrease at the rate of about 0·31′ per annum until the star is about 30′ from the pole, when it will begin to increase. The apparent place of *Polaris* is tabulated for every day of the year in the N.A.

If an observer at the equator sights *Polaris* when it is at its eastern elongation, the line of sight will be east of the true north by an amount equal to the polar distance; at the western elongation, the line of sight will be the same distance west of the true north. But if the observer is in a latitude north of the equator, and sights this star when it is at its elongation, the line of sight will make an angle with the meridian greater than the polar distance; and the farther north the observer, the greater this angle. This horizontal angle is known as the azimuth of *Polaris*, and is not to be confused with the polar distance, since the two are not the same, except at the equator.

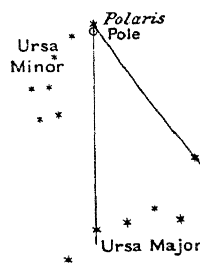


FIG. 122.

Azimuth varies with the position of the observer; *polar distance* does not. Azimuths of *Polaris* in different latitudes are tabulated in the N.A. for hour angles 0 h. to 12 h. and 12 h. to 24 h., when the star is respectively west and east of the north. Useful tables for azimuth and latitude are also given in the

N.A.(*a*). Latitude tables for extra-meridian observations are also given in "Whitaker's Almanack".

(ii) *The Pointers* (α and β *Ursae Majoris*), two well-known stars in that portion of the constellation known as *The Plough*, point directly to the Pole Star. American writers designate these "the two stars farthest from the handle of the *Great Dipper*".

Alioth (ϵ *Ursae Majoris*), "the shaft horse of Charlie's Wain", and *Mizar* (ζ *Ursae Majoris*) are familiar stars in the same constellation.

(The angular distance between two stars can be closely estimated by comparison with half the distance between the point overhead and the horizon (that is, 45°), or by the distance between the Pointers, which is roughly 5°.)

(iii) *Arcturus* (α *Boötes*) is situated about 30° on the produced curve of the tail of the Great Bear.

(iv) *Capella* (α *Aurigae*) is situated 50° from *Polaris* on a line from that star at right angles to the line of the Pointers.

(v) *Spica* (α *Virginis*) is situated 30° beyond *Arcturus* on a line from *Polaris* through the last star but one in the tail of the Great Bear.

The Great Square of Pegasus is on the Pointers' line, 60° beyond *Polaris*.

(vi) *Markab* (α *Pegasi*) is the star in the corner of the Square opposite to the Pole.

Midway between the Square and *Polaris* is the constellation *Cassiopeia* (five stars grouped like a W).

(vii) *Deneb* (α *Cygni*) is on the diagonal through the Square, produced 40° south-east to north-west.

(viii) *Vega* (α *Lyrae*), a large white star, is 25° farther along, and 10° to the right of this line.

(ix) *Altair* (α *Aquilae*) is 35° south of the line between *Deneb* and *Vega*. Situated between companions, it forms the apex of an isosceles triangle with *Deneb* and *Vega*.

(x) *Rigel* (β *Orionis*) is 65° beyond *Capella* on the line from *Polaris* through *Capella*. This line passes between ruddy *Aldebaran* (α *Tauri*) 10° west and 30° from *Capella*, and ruddy *Betelgeuse* (α *Orionis*) 10° and 40° from *Capella*. *Rigel* and *Betelgeuse* are diagonal stars in *Orion*.

(xi) *Sirius* (α *Canis Majoris*), the brightest star in the heavens, is equidistant with *Aldebaran* from a line from that star through the *Belt of Orion*.

(xii) *Procyon* (α *Canis Minoris*), which is situated 50° east of *Betelgeuse*, forms nearly an equilateral triangle with *Betelgeuse* and *Sirius*.

(xiii) *Castor* (α *Geminorum*) is 30° north of *Procyon*, and *Pollux* (α *Geminorum*) is 5° south-east of *Castor*.

(xiv) *Regulus* (α *Leonis*) forms, 45° east, the apex of an isosceles triangle with *Castor* and *Procyon*.

(xv) *Denebola* (β *Leonis*) is situated 30° eastward from *Procyon* to *Regulus*. *Denebola* forms an equilateral triangle with *Arcturus* and *Spica*.

The mean and apparent places of 207 standard stars are tabulated with relevant data in the N.A. The R.A. and declination and magnitude of 64 principal stars used in navigation are given on pp. 1–2 of the month in the N.A. (*a*) where also are tabulated the apparent places of 172 stars, the values being given to 0·1′ of arc and 1 s. of time. Whitaker gives a concise table of the mean positions of a selection of stars visible at Greenwich on January 1.

Planets. Besides the fixed stars, the planets Venus, Mars, Jupiter, and Saturn are often conspicuous celestial bodies. These must never be mistaken for stars. Their positions can be found for any date in the "Nautical Almanac", and so no difficulty should arise in locating them among the constellations at the time of observation.

(III) **Computations.** Chambers' Seven-figure Mathematical Tables are most commonly used in geodetic astronomy, the more comprehensive compilations, as used on the Continent, being more especially adapted to precise calculations in geodesy. On the other hand, sufficient accuracy for many field operations can be obtained with six-figure logarithmic trigonometrical functions in conjunction with five-figure logarithms.

A prescribed form for recording the observations and making the calculations is most desirable in the present connection, and an excellent selection will be found in "Topographical Surveying" by Sir Charles Arden-Close. Systematic procedure in computing is the keynote of speed and accuracy, and practice in obtaining simplicity and clarity in placing and transposing numerical data is essential in connection with all geodetic observations. It is impossible to embody the complete calculations relative to selected problems in a text-book of this nature, and the example of p. 300 is inserted as suggestive of the manner in which the calculations should be made.

The general case in field astronomy involves the solution of what is sometimes styled the "Astronomical Triangle" *SPZ*, *S* being the sun, a star or a planet.

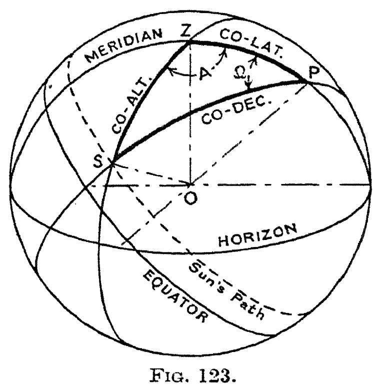


FIG. 123.

Three quantities must be known in this case of an oblique spherical triangle (Fig. 123), and two in problems introducing right-angled spherical triangles, as in the case of celestial bodies at elongation or on the prime vertical.

The *observed* quantities are invariably the altitude and the time of

observation, and in certain problems the horizontal angle from a referring point.

The *data* may consist of the latitude, the longitude, and the sidereal time at the previous midnight (or noon) at the basic meridian, Greenwich usually.

The *reduced* quantities are the tabular declination at the instant of observation, the right ascension, and, in solar observations, the equation of time.

Most of these have been considered, though opportunely a note may be inserted on the subject of solar declination and right ascension.

Solar declination and right ascension. *The declination of the sun at the instant of observation must be found by reduction for local mean time and longitude to the corresponding time at the meridian for which the solar ephemeris is prepared, usually an observatory, which is Greenwich in the case of the Nautical Almanac.*

No difficulty arises if the times are noted on a chronometer or clock keeping G.M.T., while if the S.M.T. of a standard meridian is kept, the correction will be for an integral number of hours, which embodies the longitude.

Otherwise, the reduction to Greenwich time is less difficult than it first appears to be.

For local mean time with a watch, the maximum error of declination for an error of 1 hour or 15° of longitude on L.M.T. is less than 1 minute of arc.

The sun's apparent declination and R.A. at G.M.M. are tabulated for each day in the N.A., and corresponding tables are given for G.A.N. In the N.A. (*a*) these values are given in two-hourly periods of each day, accounted from G.M.M. Whitaker gives the apparent R.A. and declination at G.M.N. with hourly variations on every day. Of 208 Fundamental stars (excepting the R.A. of *Polaris*) the declination varies from 0 s. to 20·12 s. and the R.A. from 0·015 s. to 6·349 s. in the course of a year.

The sign of the declination is *plus* or *minus* according as the declination is prefixed with the *same* letter as, or the *opposite* to, that denoting the hemisphere of observation, N. or S., as the case may be.

The true or actual sun moves in a great circle known as the ecliptic, which is inclined with an *obliquity* of 23° 27′ to the celestial equator, cutting the latter in two diametrically opposite points known respectively as the First Points of Aries and Libra. Hence the declination of the sun is constantly changing, being 0° at the equinoxes and 23° 27′ N. or S. at the solstices.

The Right Ascension of the sun does not increase uniformly for the following reasons :

(*a*) The earth's path round the sun is an ellipse with the sun at one of the foci, and its motion varies in such a way that a line to the sun

sweeps out equal areas in equal times, in accordance with the law of gravitation.

(*b*) Apart from the fact that the sun's motion in the ecliptic is not uniform, Right Ascensions are measured along the celestial equator, and not along the ecliptic.

The variations due to these deviations from the motion of the mean sun are known respectively as eccentricity and obliquity, and the net effect is the difference between the Right Ascension of the mean and true suns, and is conversely the equation of time.

Example†. Solar observations are to be made at a station in Latitude 42° 20′ 00″ N. and Longitude 8 h. 11 m. 45 s. W. on Feb. 12, 1935, the following data being available :

Declination (G.A.N.) S. 13° 54′ 00″, decreasing 0·83′ per hr.

Equation of Time (G.M.N.) 14 m. 22·7 s., decreasing 0·03 s. per hour and subtractive from mean time.

Calculate (*a*) the sun's altitude at 9.50 a.m., Pacific Time (120th); (*b*) the standard mean time of sunrise, as assessed by the sun's centre.

(*a*) G.M.T. of Obs., 9 h. 50 m. + 8 h. = 17 h. 50 m., or 5 h. 50 m., p.m. = 5·833 h.

G.A.T. of Obs. 5 h. 50 m. − (14 m. 22·7 s. − 0·03 × 5·83) = 5 h. 35 m. 37·5 s. p.m.

At G.A.T., 5 h. 35 m. 37·5 s., declination = 13° 54′ − 5·594 × 0·83′ = declination at 17 h. 35 m. 37·5 s. − 8 hr. 11 m. 45 s. = 9 h. 23 m. 52·5 s. L.A.T., which is an hour angle of 21 h. 23 m. 52·5 s. = 320° 58′ 07″.

$$\cos SZ = \cos PZ \,.\, \cos SP + \sin PZ \,.\, \sin SP \,.\, \cos SPZ, \text{ which reduces to}$$

$$\sin\alpha \;\; = -\sin\lambda\sin\delta + \cos\lambda\cos\delta \,.\, \cos \Omega \quad (\sin\delta -;\; \cos\delta +)$$

$$= -0{\cdot}1608986 + 0{\cdot}5576115 = 0{\cdot}3967129\,;\text{ or } \alpha = 23°\,22'\,22{\cdot}25''.$$

$$(b)\ \sin\alpha = 0\,;\quad \cos SPZ = \frac{\sin\lambda\sin\delta}{\cos\lambda \,.\, \cos\delta},\ \text{giving } SPZ = 77°\,56'\,50{\cdot}5'',\ \text{with}$$

$$\Omega = 282°\,03'\,9{\cdot}5'' = 18\text{ h. }48\text{ m. }12{\cdot}6\text{ s., or }6\text{ h. }48\text{ m. }12{\cdot}6\text{ s. L.A.T.}$$

	h.	m.	s.
G.A.T. = 6 h. 48 m. 12·6 s. + 8 h. 11 m. 45·0 s. =	14 h.	59 m.	57·6 s.
Adding E.T. 14 m. 22·7 s. − 0·09 s.		14	22·6
G.M.T.	15	14	20·2
Long. W.	8	00	00·0
S.M.T. of sunrise	7	14	20·2

QUESTIONS ON ARTICLE 1

1. An observation of the sun at 4 p.m. G.M.T. gave the following results tabulated below.

Object	Face	Altitude		Level	
		Micro *A*	Micro *B*	Eye end	Object end
$\underline{O}$ -	*L*	58° 47′ 20″	47′ 25″	13	14
$\overline{O}$ -	*R*	59° 19′ 45″	19′ 35″	15	12

Find the corrected altitude if the refraction correction = 57 tan z seconds, the sun's horizontal parallax is = 8·71 seconds, and the value of one division of the altitude level is 8 seconds of arc. z is the coaltitude. (I.C.E.)

[59° 3′ 02·57″.]

2. What are "parallax" and "refraction" and how do they affect the measurement of vertical angles in astronomical work?

Give rough values of the corrections necessary when measuring a vertical angle of 45°. (I.C.E.)

[+6″; −57″.]

3. State the instrumental and other corrections which must be applied to the vertical circle readings of a theodolite, when the altitude of the sun is observed in connection with a position or azimuth determination. Explain briefly the reasons for each correction. (I.C.E.)

4. What do you understand by the following terms:

Aphelion, solstice, equinox, sidereal time, right ascension, hour angle, and ecliptic? (T.C.C.E.)

5. What are the systems of coordinates employed to locate the position of a heavenly body? Why is it necessary to have several systems instead of one? Illustrate your answers by sketches where necessary. (T.C.C.E.)

6. Draw a diagram to show the celestial sphere for a point 15° N., 75° E., showing the horizon, meridian, zenith, pole, and celestial equator.

Mark also the path of the sun at midsummer, and the position of α Bootes (decl. 20° N., R.A. 14 hr. 10 min.) at 22 hrs. G.S.T. (T.C.C.E.)

ARTICLE 2: TIME

Apparent time. Apparent or solar time is the hour angle of the sun reckoned westwards from the meridian. Commonly expressed, it is time as determined by the sun and as indicated on sundials. **Apparent noon** is the instant at which the sun's centre crosses the meridian; an instant, in

general, not identical with noon as recorded by watches and clocks, since the sun's apparent place in the heavens is constantly changing on account of the earth's orbit. Thus clocks set to keep apparent time would need regulating every day; and hence the expedient of *mean*, or average, time.

The **equation of time** is the difference between apparent and mean time. Formerly apparent time was determined by solar observation, and was reduced to mean time by means of this equation. Astronomers, however, reduce mean time from sidereal time, and today navigators employ wireless signals. Hence the "Nautical Almanac" merely shows the correction to be applied to mean time to give apparent time, the values being tabulated in the sense G.A.T.–G.M.T., and are to be added algebraically to mean time to give apparent time. In problems involving the Equation of Time for a given apparent time, use is made of the tables showing the G.M.T. of the sun's transit.

The Equation of Time (E.T.) varies from 0 to about 16 minutes at different seasons of the year. Its values are sometimes prefixed with the *plus* and *minus* signs, or are specified as "sun after clock" and "sun before clock", indicating that such values are to be added to, or subtracted from, the apparent time in order to reduce it to mean time, civil or astronomical, which are now identical in that both are reckoned from midnight.

Thus at local apparent noon on two days when the equation of time at local mean noon is 5 m. 6 s. and −11 m. 3 s., the corresponding mean times are respectively 12 h. 5 m. 6 s. and 11 h. 48 m. 57 s. in both astronomical and civil time. Formerly these would have been respectively 0 h. 5 m. 6 s. and 23 h. 48 m. 57 s. astronomical time and 12 h. 5 m. 6 s. p.m. and 11 h. 48 m. 57 s. a.m., the latter astronomical time being of the date previous to the day of the corresponding value in civil time.

In the "Abridged Nautical Almanac" (N.A.(*a*)) the equation is tabulated as E = 12 h. − E.T. on pages III-IV of the month for G.M.T., 0 h. to 24 h. daily. Formerly the values were tabulated in the N.A. on page I of the month for G.A.N. and on page II of the month for G.M.N., the former column serving more particularly for converting apparent time into mean time and the latter mean time into apparent time at the same meridian. Many surveyors preferred this arrangement; "Whitaker", however, gives the equation at G.M.N. as "Add to" or "Subtract from" mean time, and when a break occurs in a column it signifies that a change of precept occurs in the course of the month.

Astronomical mean time. Up to midnight, Dec. 31, 1924, the times styled G.M.T. in the "Nautical Almanac" differed from civil mean time only in the fact that the civil day began at midnight whereas the astronomical day was accounted to 24 hours from the following noon. Hence the rule of adding 12 hours to local mean times of forenoon observation in

order to obtain astronomical mean times of observations prior to Jan. 1, 1925. Now both the astronomical and civil day begin at midnight, introducing the term, **Greenwich Mean Midnight**. The International Astronomical Union recommended that **Universal Time** (U.T.) should be used internationally for G.M.T. reckoned from midnight, and that in future Greenwich Civil Time (G.C.T.) should not be used, the objection to this being that the term has a doubtful meaning in Britain and in summer might imply " Summer Time ".

Local mean time. Civil mean time and local mean time are synonymous terms in so far as they denote time in its relation to everyday life. Local mean time, the more common term, at once indicates that mean time at the place of observation is to be understood in the case of appreciable departure from the longitude of the nearest ruling meridian. " Summer Time " and its extension are merely social and economic expedients, and observations recorded thus must be reduced to astronomical mean time.

Standard mean time. Since 1883, the system of standard times by zones has been gradually adopted, and now in most parts of the world a standard mean time is used, differing by an integral number of hours either fast or slow of Greenwich Mean Time. Standard times generally for most parts of the world are given in both editions of the " Nautical Almanac ", while " Whitaker " gives a classified list of places where the hourly zone system is used, adding certain places with half-hourly differences. In North America mean time is controlled by five **standard meridians**, so that it admits of no variations except those of even hours, or 15° of longitude. The zone, or belt, extends $7\frac{1}{2}°$ on either side of these meridians, and in relation to overlapping for the convenience of railroad systems, it is necessary to style the times appropriately, as in the following summary, which defines the meridian west and the hours of mean time slow of Greenwich Mean Time.

Atlantic Time :	60th Meridian (near Hopedale, Labrador) (formerly Maritime or Intercolonial)	−4 hr.
Eastern Time :	75th Meridian (near Philadelphia)	−5 hr.
Central Time :	90th Meridian (near St. Louis)	−6 hr.
Mountain Time :	105th Meridian (near Regina, Saskatchewan)	−7 hr.
Pacific Time :	120th Meridian (near Sacramento, California)	−8 hr.

These meridians and their belts apply in part to South America, but Brazil is divided into three zones in which the times are 3, 4, and 5 hours slow on Greenwich Mean Time. In the U.S.S.R. zones from 3 hours to 13 hours fast have been adopted. The true L.M.T. at any place will be faster or slower than the standard time according as the place is east or west of the standard meridian. Thus the true local mean time at Boston, Massachusetts, is about 16 minutes fast on standard time, while the local

and standard times nearly agree at Denver, Colorado, which is almost on the 105th meridian.

Sidereal time. Sidereal time is the perfectly regular measure of time, of which 24 hours is the time of a complete revolution of the earth about its axis with respect to any fixed star, which, if visible to the naked eye, may be 300 light years from the earth, a light year being the distance light travels in a year, or approximately 63,000 times the distance of the earth from the sun. The actual *mean time* in which the complete revolution is performed, however, is 23 h. 56 m. 4·091 s. Thus 24 hours of *mean time* are equal to 24 h. 3 m. 56·555 s. of *sidereal time* ; and if a star crosses the meridian on successive evenings, its passage will be 3 m. 55·909 s. of mean time earlier each evening. Gaining thus about 4 minutes per day, sidereal time would be of no use as civil time, for, unlike the latter, it is independent of the sun's daily motion, which controls the phases of everyday life. Thus, in keeping with sidereal time, the surveyor would have to commence duties about 4 minutes later each successive day, in order to avoid commencing at midnight later. On the other hand, he requires sidereal time in various stellar observations, particularly in correcting watches and chronometers. Since sidereal clocks keep *mean* sidereal time, this is to be distinguished from *apparent* sidereal time as determined by observations, the difference being the nutation in R.A. analogously with the E.T. and mean and apparent solar times accordingly.

If the earth be imagined to rotate about its axis in the counter-clockwise direction while revolving likewise in a circular orbit about the sun at the centre, and on a certain day a meridian of the earth M be assumed to be in line with the sun S at a finite distance and a star N at an infinite distance ; then the next day the earth will have moved to a position M' so that the meridian passes the star on a parallel $M'N'$ before it reaches the sun, and the angle MSM' will measure the difference between a sidereal day and a solar day.

Sidereal time commences when the " First Point of Aries ", the zero of right ascensions, is on the meridian of the place of observation, that is, at 0 h. or 24 h. of sidereal time, or " sidereal noon ".

Formerly the " Mean Time of Sidereal Noon " was given in the N.A. as the " Mean Time of Transit of the First Point of Aries ". Now it is tabulated therein as " Apparent Sidereal Time " at G.M.M. In the " Abridged N.A.", the values tabulated instead of " Right Ascension of Mean Sun " (which is G.S.T.) are styled " R ", R being R.A.M.S. $\pm$12 h. At G.M.N., G.S.T. = R.A.M.S. = R + 12 h., but at G.M.M., G.S.T. = R.A.M.S. + 12 h. = R, which are the values given, the two-hourly intervals really showing the changes from the mean midnight values of R. Thus on Table VI, Oct. 26, 1940, R is 2 h. 16 m. 51·6 s. at 0 h. G.M.T. and 2 h. 18 m. 49·9 s. at 12 h. G.M.T., the difference being for 12 h. at 0·96 sec./hr. G.S.T. at G.M.N. and G.M.T. at 0 h. G.S.T. are given in

"Whitaker's Almanack", the values being to the nearest second compared with exact values in the "Nautical Almanac".

The local sidereal time (at any place) is the hour angle of the First Point of Aries (♈), and is the Right Ascension of a star plus its hour angle; and when a star is at upper transit (or culmination) the local sidereal time is the right ascension, the hour angle of the star being zero. The sidereal interval (S.I.) between mean noon and a star's culmination is therefore equal to its right ascension less local sidereal time at mean noon. If this interval is converted into mean time, this is the mean time interval between local mean noon and the star's culmination. The relation between Greenwich and any local sidereal time is as follows:

Let it be G.M.N. and let the G.S.T. be 15 hours $= \omega$ ahead of this. Then at a place in long. $L°$ west of Greenwich, the L.M.T. will be $L°/15$ hrs. slow, but by the time it is L.M.N. in long. $L°$ west, the angle will have increased to $\omega + (9{\cdot}86\ L°/15$ sec.). Similarly for a place in long. $L°$ east, L.M.N. will occur when it is $L°/15$ mean time hours before G.M.N., and the angle ω will be reduced to $\omega - (9{\cdot}86\ L°/15$ sec.). Hence the rule:

$$\text{L.S.T., L.M.}\begin{Bmatrix}\text{N}\\\text{M}\end{Bmatrix} = \text{G.S.T. of G.M.}\begin{Bmatrix}\text{N}\\\text{M}\end{Bmatrix} \pm 9{\cdot}86\text{ s. per hr. }\frac{L°}{15}\text{ of long.}$$

Formulae showing the relationship between L.M.T., L.S.T., L.A.T., etc., might be readily derived, but in general these should be avoided by working from fundamental relationships, aided if necessary with diagrammatic sketches.

Conversion tables. Tables VII, VIII of the N.A. are for reducing arc to time and time to arc, while IX, X are for converting minutes and seconds of arc to decimals of a degree, and vice versa. Chambers gives abbreviated tables for reducing degrees to time and time to degrees, the values corresponding to H., M., S., and T. ("thirds", or $\frac{1}{60}$ sec.).

Tables III, IV, and V of the N.A. are for converting mean solar intervals into sidereal time and vice versa. Abbreviated tables for the purpose will be found in "Whitaker's Almanack" and in "Chambers' Tables". In the absence of these the following mutual relations may be used:

L.M.T. to L.S.T. and L.S.T. to L.M.T.

			Hour	Min.	Sec.
		per Solar			
Add	in sec.	-	9·8565	0·1642	0·0027
Subtract	,,	-	9·8296	0·1638	0·0027
		per Sidereal			

The following is a series of illustrative examples, introducing the various modes of calculation and embodying the changes in terminology that have occurred since 1925.

EXAMPLES ON TIME

(1) State the L.M.T. and the S.M.T. at Perth (W.A.) at 4.25 p.m. G.C.T. on Jan. 15, 1920, the longitude of the place being 115° 50′ 25·5″ E.

			h.	m.	s.
G.C.T., 4·25 p.m. = G.M.T. on date			4	25	0·0
115° long. to time,	115 × 4 m.	7 h. 40 m. 00·0 s.			
50′ ,, ,,	50 × 4 s.	03 20·0			
25·5″	$=\frac{1}{15}(25{\cdot}5)$ s.	00 01·7			
115° 50′ 25·5″	Fast	7 43 21·7	7	43	21·7
Perth Jan. 16, L.C.T., 0 h. 8 m. 21·7 s. a.m. = L.M.T., Jan 15			12	08	21·7

S.M.T. will be that of the 120th meridian (W.A.Time),
Jan. 16, 0 h. 25 m. 00 s. a.m.

(2) State the L.M.T. and the error on S.M.T. at New York in long. 4 h. 55 m. 53·64 s. W., at G.A.T. 23 h. 35 m. 10 s. on Dec. 2, 1924, the E.T. (G.A.N.) being 10 m. 31·33 s., subtractive from A.T., and decreasing 0·965 s. per hr.

		h.	m.	s.
G.A.T., Dec. 2		23	35	10·00 = 23·592 h.
E.T. (G.A.N.)	10 m. 31·32 s.			
Decrease 0·965 s. for 23·592 h.	22·76			
Sub. from A.T.	10 08·56	−10		08·56
G.C.T., Dec. 3, 11 h. 25 m. 01·44 s. a.m.		= 23	25	01·44 G.M.T., Dec. 2.
	Long. West	− 4	55	53·64
New York, C.M.T., Dec. 3, 6 h. 29 m. 7·8 s. a.m.		= 18	29	07·80 L.M.T., Dec. 2.

S.M.T. was that of the 75th meridian, 5 h. slow of G.M.T., or 6 h. 25 m. 01·44 s. a.m. *Eastern Time.*

(3) State the L.A.T. of an observation at Madrid in long. 14 m. 45·09 s. W. at L.M.T. by chronometer 10 h. 14 m. 20 s. on May 17, 1927, the E.T. at G.M.N. being 03 m. 44·52 s. subtractive from A.T. and decreasing 0·046 s. per hr.

		h.	m.	s.
G.M.T. of Obs. = 10 h. 14 m. 20 s. + 14 m. 45·09 s.		= 10	29	05·09
M.T. Interval before G.M.N. = 1·5152 hrs. = 1 h. 30 m. 54·91 s.				
E.T. (G.M.N.)	3 m. 44·52 s.			
Increase for 1·5152 h. at 0·046 s./hr.	00·07			
Add to G.M.T.	3 m. 44·59 s.		03	44·59
G.A.T. of observation		10	32	49·68
Long. West			14	45·09
L.A.T. of observation		10	18	04·59

(4) The apparent time by solar observation at Tokyo in long. 139° 44′ 30·3″ E. was 10 h. 24 m. 6 s. on Aug. 4, 1930, the mean time chronometer reading 10 h. 30 m. 5 s. accordingly. Find the error of the chronometer, given that the E.T. (G.M.N.) on that date was 6 m. 0·70 s. additive to apparent time and decreasing 0·20 s. per hr.

					h.	m.	s.
L.A.T. of observation					10	24	6·00
		h.	m.	s.			
139° Long. to Time	139° × 4 m.	9	16	00·00			
44′	44′ × 4 s.		2	56·00			
30·3″				02·02			
139° 44′ 39·3″	Fast	9	18	58·02	9	18	58·02
	G.A.T. by Obs. = 1·0855 h.				1	05	07·98
E.T. (G.M.N.)		6 m. 0·70 s.					
Decrease for 1·0855 h. at 00·20 s./h.			0·22				
		6	0·48			06	00·48
		G.M.T. of Obs.				11	08·46
		Long. East				18	58·02
		L.M.T. of Obs.			10	30	06·48
		by chronometer			10	30	05·00
		Chronometer slow					01·48

(5) Determine the L.S.T. corresponding to 9·35 p.m. at Madras in long. 5 h. 20 m. 59·62 s. E. on July 15, 1933, the G.S.T. at G.M.N. being 7 h. 35 m. 29·6 s. State the hour angle of β Ursae Minoris at this C.M.T., the star's right ascension being 14 h. 50 m. 55·7 s.

					h.	m.	s.
G.S.T. (G.M.N.)					7	35	29·60
Long. East, 5 h. 20 m. 59·62 s.							
5 × 9·856 = 49·28 s.							
20 × 0·164 = 3·28							
60 × 0·0027 = 0·16							
52·72 s.				Retardation			−52·72
				L.S.T. (L.M.N.)	7	34	36·88
C.M.T.	9 h. 35 m. 00 s. p.m.						
9 × 9·856 = 88·70 s.							
35 × 0·164 = 5·74							
94·44 s.		01	34·44	Acceleration			
S.I. Int. from L.M.N.	9	36	34·44		9	36	34·44
				L.S.T.	17	11	11·32

Hour angle of star = 17 h. 11 m. 11·32 s. − 14 h. 50 m. 55·7 s.

= 2 h. 20 m. 15·62 s.

(6) State the L.M.T. corresponding to L.S.T. 20 hr. 42 m. 10 s. at Vienna in long. 1 h. 5 m. 21·35 s. E. on May 31, 1926, the G.S.T. at G.M.N. being 4 h. 31 m. 55 s.

					h.	m.	s.
L.S.T.					20	42	10·00
G.S.T. (G.M.N.)	4 h. 31 m. 55·00 s.						
Retardation for Long. East							
(Ex. Chambers') 9·86 + 0·82 + 0·06 s.			– 10·74				
L.S.T. (L.M.N.	4	31	44·26		4	31	44·26
				S.I. from L.M.N.	16	10	25·74
Reducing to M.T. (Ex Chambers') :							
2 m. 37·27 s. + 1·64 s. + 0·07 s.						– 02	38·98
				M.T. Interval from L.M.N.	16	07	46·76
L.M.T. June 1, 4 h. ·07 m. 46·76 s.							

(7) Find the L.A.T. corresponding to L.S.T. 6 h. 22 m. 16 s. by chronometer near Prague in long. 0 h. 57 m. 01·70 s. E. on April 12, 1940, the G.S.T. (G.M.N.) being 1 h. 29 m. 9 s., and the E.T. (G.M.N.) 0 m. 49 s., subtractive from M.T. and decreasing 0·65 s. per hr.

					h.	m.	s.
L.S.T.					6		16·00
G.S.T. (G.M.N.)	1 h. 29 m. 9·00 s.						
Retardation for Long. East							
(Ex. Whitaker)			9·35				
L.S.T. (L.M.N.)	1	28	59·65		1	28	59·65
				S.I. from L.M.N.	4	53	16·35
Correcting to M.T. (Ex. Whitaker)							48·04
				L.C.T. = M.T. Int. from L.M.N.			28·31
				Long. East		57	01·70
				Corresponding G.C.T. = 3·924 h. p.m. =	3	55	26·61
E.T. (G.M.N.)	0 m. 49·00 s.						
3·924 h. at 0·65 s./hr.	02·55	decreasing					
	46·45 s.						– 46·45
				G.A.T. 12 h. +		54	40·16
				Long. East		57	01·70
				L.A.T. 12 h. +	4	51	41·86

(8) Determine the L.M.T. of transit of Polaris at Quebec in long. 4 h. 44 m. 49·38 s. W. on Oct. 26, 1940, the R.A. of the star being 1 h. 44 m. 46·8 s., and R, 2 h. 18 m. 49·9 s. at 12 h. G.M.T.

				h.	m.	s.
L.S.T. of Transit = (R.A. + 24 h., since S.T. > R.A.)				= 25	44	46·8
G.S.T. (G.M.N.) = R.A.M.S. = R + 12 h. (Ex. N.A. (a)),	14 h.	18 m.	49·90 s.			
Correcting for Long. West			46·80			
L.S.T. (L.M.N.)	14	19	36·70	14	19	36·7
S.I. from L.M.N.				11	25	10·1
Correcting to M.T.					− 1	52·2
M.T.I. from L.M.N.			p.m.	11	23	17·9
L.M.T.				23	23	17·9

(9) Determine the hour angles of the (a) *Mean Sun* and (b) the *True Sun* at Basle in long. 0 h. 30 m. 20·2 s. E. at 3.30 p.m. on Aug. 1, 1940, the abridged N.A. being available.

(a) ♌ = L.S.T. − R.A., where ♌ is the hour angle;
= L.S.T. − R.A.M.S.;
but L.S.T. = L.M.T. + R.A.M.S. ± 12 h.
= L.M.T. + R, R being tabulated;

	h.	m.	s.
G.M.T. = 15 h. 30 m. 00 s. − 30 m. 20·2 s.	= 14	49	39·8
At G.M.T., 14 h. 49 m. 39·8 s.,			
R = 20 h. 40 m. 06 s. (Ex. N.A.(a)) $+\dfrac{49\cdot66}{120}(9\cdot85)$ s.	= 20	40	10·1
	35	29	49·9
At G.M.T., 12 h., R + 12 h. = R.A.M.S. = G.S.T. (G.M.N.)	32	39	46·3
♌ =	2	50	03·6

(b) Since the R.A. of the true sun is not given in the N.A. (a),
. = ♌ ± 12 h.
= L.M.T. − E.T.;

= L.M.T. + E, with E as tabulated.

	h.	m.	s.
L.M.T. =	14	49	39·80
= (11 h. 53 m. 48·60 s. +	11	53	48·72
, =	2	43	28·52

(10) The rhythmic signal from Rugby at G.M.T. 10 h. was observed at coincidence with 23 h. 11 m. 8·5 s. L.S.T. on a sidereal clock on May 31, 1940. Find the longitude of the station and the L.M.T., given that G.M.T. at 0 h. G.S.T. was 7 h. 25 m. 25·3 s.

				h.	m.	s.
G.M.M.				24	00	00·00
G.M.T. of Transit of ♈	=7 h.	25 m.	25·3 s.			
Acceleration		01	13·2			
S.I. from G.M.M.	7	26	38·5	7	26	38·50
			G.S.T. (G.M.M.)	16	33	21·50

	h.	m.	s.
L.S.T. of Coincidence	23	11	08·50
G.S.T. (G.M.M.)	16	33	21·50
S.I. from G.M.M.	6	37	47·00
Retardation		01	05·20
Long. East = M.T. Int. from G.M.M.	6	36	41·80
Adding G.M.T.	10	00	00·00
L.M.T.	16	36	41·80

OBSERVATIONS FOR TIME

First method. (*a*) **By meridian transit of the sun.** The basis of this method is the fact that the sun's centre crosses the meridian at local apparent noon, and that the difference between the time of this transit and local mean noon is equal to the Equation of Time for the time of observation.

The field work, therefore, consists in clamping the telescope of the instrument in the meridian, and observing the time of transit of the sun's centre, which is the mean of the times of transit of its right and left limbs.

Since the Equation of Time is tabulated for Greenwich midnight (noon), it is evident that it cannot be exact at any other place unless the longitude is known. But at the present stage the longitude is not known presumably, and it therefore remains to ascertain how closely it needs to be approximated.

In the first place, the maximum variation in the equation is only $1\frac{1}{4}$ seconds per hour, and at times only a small fraction of a second. Hence $1\frac{1}{4}$ seconds of time represents a maximum error in the equation for an error in longitude of 15°, which corresponds to more than a thousand miles at the equator.

Now, in these days it is unlikely that the surveyor will be at any place on land where he is unable to ascertain his position to within 100 miles of some place indicated on an ordinary map, and even this error in distance

would lead to no greater error in the equation than $\frac{1}{3}$ of a second in as extreme a latitude as 70°.

But these errors are inappreciable in comparison with those of observation with ordinary theodolites, the work involving the following requirements : (1) a telescope more powerful than those of ordinary theodolites ; (2) accuracy in the vertical adjustment of the theodolite ; (3) accuracy in meridian employed ; (4) exact means of determining the times of transit. Besides, solar transits should not be observed when the sun souths very near the zenith.

Hence the method is often unsatisfactory and, in good work, should be superseded by extra-meridian observations, which are always dependable, so far as the surveyor is concerned, though at times inapplicable on account of lack of data.

(*b*) **By transit of a star.** The foregoing method is simplified if the transit of a star instead of the sun is observed, though the limitations of an ordinary theodolite are imposed, and the transit is determined by equal altitudes, as in the succeeding method. On the other hand, the method is that employed by astronomers through the medium of a transit instrument, the collimation of which is fixed in the meridian. Normally the instant at which a known star crosses the central hair is observed, for at this instant the local sidereal time is equal to the right ascension of the star ; hence if the clock is keeping sidereal time, its error is immediately known, while if it is keeping mean time, the local mean time is reduced from the longitude and Greenwich sidereal time at mean midnight.

Although the transit of a single star gives more accurate results than any theodolite observation, the accuracy is enhanced by observing eight to ten stars, reversing the telescope in its pivots at least once during the observations.

Second method. (*a*) **By equal altitudes of the sun.** Solar transits may be obviated by observing the sun at equal altitudes, the same edge of the sun's image being brought to the horizontal hair and the image bisected by the vertical hair of the diaphragm. The approximate values of the latitude and Greenwich mean time must be known, since the following correction must be applied for the change in the sun's declination during the observations. The afternoon hour angle will be greater than the forenoon hour angle for the same altitude by

$$\Big(\sin \Omega \quad \tan \Omega\Big)$$

where $\Delta\delta''$ is the increase in the declination in seconds of arc and Ω is half the mean time interval, also in arc. The average hour angle, therefore, will be increased by half this amount, and a correction of $-\left(\frac{k}{15 \times 2}\right)$ sec. must be applied to the average time of the two altitudes, the minus sign

indicating that the declination is increasing. The reduction of mean time is then similar to that described for the First Method (*a*).

(*b*) **By equal altitudes of a star.** The preceding method is simplified if a known star is observed, since no knowledge of the latitude or declination is involved ; though, on the other hand, a long period must elapse between the observations, and during this interval the changes in the refractive conditions of the atmosphere may introduce errors. However, the interval can be reduced by selecting a star with a declination not much less than the latitude ; though if the declination be equal to the latitude, the star will pass through the zenith, necessitating the use of the diagonal eyepiece. It is essential that the star should be distant from culmination when observed, so that its altitude changes rapidly ; and hence a star near the prime vertical should be observed.

The actual observation is simpler than in the case of the sun, though, likewise, the face of the theodolite must not be changed with respect to a pair of altitudes. The time should be recorded for the instant the image of the star crosses the intersection of the cross-hairs at the same altitude, the average being the watch time of upper or lower transit, as the case may be. Whenever possible two or four pairs of equal altitudes should be observed on the same star, the altitude read and altered by about 30″ for the next observation, and the vertical circle reset to the corresponding altitude in the opposite order.

When a star is observed on the prime vertical, the formulae are those of a spherical triangle with a right angle at the zenith Z :

$$\cos \mathfrak{h} = \cos SPZ = \frac{\cot PS}{\cot PZ} = \frac{\tan \delta}{\tan \lambda};$$

$$\sin \alpha = \frac{\cos PS}{\cos PZ} = \frac{\sin \delta}{\sin \lambda};$$

α being the altitude, $\mathfrak{h}$ the hour angle, λ the latitude, and δ the declination.

Third method. (*a*) **By one extra meridian observation of the sun.** Consider the *solar hour angle* SPZ of the spherical triangle discussed in connection with the Fourth Method of determining meridians (p. 296).

Here the problem is that of solving that triangle for the hour angle SPZ, given the *three sides* :

PZ, the co-latitude $= 90° - \lambda$;

PS, the co-declination $= 90° - \delta$

and ZS, the co-altitude $= 90° - \alpha$.

$$\frac{\sin (s - PS) \, . \, \sin (s - PZ)}{\sin s \, . \, \sin (s - ZS)}$$

where s is the semi-sum of the three sides.

Or $2 \log \tan \frac{1}{2}SPZ$

$$= \log \operatorname{cosec} s + \log \operatorname{cosec}(s - ZS) + \log \sin(s - PS) + \log \sin(s - PZ).$$

The hour angle reduced to time is the apparent time of the observation; that is, the difference in apparent time between the time of observation and the time of culmination, being before or after apparent noon according as the sun is observed east or west of the meridian. This apparent time is then reduced to mean time by applying the Equation of Time, duly corrected for longitude, as already explained. The error of the watch on local mean time is the difference between the time of observation by watch and the time of observation as determined by the calculations.

The respective errors in time, expressed as hour angle Ω, due to errors in altitude ($d\alpha$), in latitude ($d\lambda$), and in declination ($d\delta$) will follow by differentiating the formula:

$$\cos SPZ = \cos \Omega = \frac{\sin \alpha - \sin \delta \, . \, \sin \lambda}{\cos \delta \, . \, \cos \lambda}.$$

$$d\Omega_1 = \frac{d\alpha}{\sin SZP \, . \, \cos \delta}; \quad d\Omega_2 = \frac{d\lambda}{\tan SZP \, . \, \cos \lambda}; \quad d\Omega_3 = \frac{d\delta}{\tan PSZ \, . \, \cos \delta},$$

where SZP is the azimuth of the celestial body and PSZ its parallactic angle.

(*b*) **By an extra meridian observation of a star.** A sidereal hour angle may be determined in like manner from a fixed star, preferably one easterly or westerly, situated about midway between horizon and meridian. Here the angle SPZ reduced to its time equivalent is an interval of sidereal time, and this is to be reduced to its mean time equivalent by the method explained on pp. 277, 281.

Fourth method. By wireless time signals. The universal development of wireless telegraphy reduces the use of time signals to the simplest and, next to triangulation, the most accurate method of determining longitude.

Wireless time signals are transmitted from numerous stations, giving signals from which G.M.T. can be determined in most parts of the world. As is well known, British stations transmit G.M.T. by a system of six 1-second dots, the beginning of the last dot being considered the actual signal. Normally the majority of accurate signals are correct to at least 0·05 sec., being operated by precise mechanism connected with the standard clock of an observatory. Whenever accuracy as high as to 0·01 sec. is required, the *rhythmic* signal is the most suitable.

Example†. An altitude observation of the sun was made with a 7″ nautical sextant and artificial horizon on June 5, 1923 at Queen Mary College, in lat. 51° 31′ 50″ N. and long. 0° 02′ 45″ W. A single altitude was observed as 43° 02′ 15″ at 4 h. 11 m. 05 s. (G.S.M.T.) by watch. Immediately afterwards, the direct and reflected images were brought

into contact, above and below, giving the following readings: (*a*) 31′ 45″ on arc; (*b*) 31′ 25″ on arc of excess.

Determine the error of the watch, using the following data:

Refraction, 57″ cot α; parallax in altitude, 8·7″ cos α; α being the apparent altitude of the sun.

Sun's declination, G.M.N., on date, N. 22° 27′ 46·2″, with an *additive* hourly variation of 17·25″.

Equation of Time, G.A.N. on date, 1 m. 51·9 s., *subtractive*, with a decreasing hourly variation of 0·424 s.

Mean observed altitude		43° 02′ 15″
Subtract ½ index error for this half angle		05″
Apparent altitude		43 02 10
Refraction corr., 57″ × cot (43° 2′ 10″)	−01′ 01″	
Parallax, 8·7″ cos (43° 2′ 10″)	00′ 06″	−55″
True altitude		43 01 15

Approx. G.M.T. at instant of observation, 3 h. 11 m. 05 s. = 3·184 h., which is interval from previous G.M.N.

Sun's declination at G.M.N., on date	N. 22° 27′ 46·2″
Correction for 3·184 h. at 17¼″	54·9
Declination at instant of observation	N. 22° 28′ 41″

Co-declination, SP =	67° 31′ 19″		
Co-altitude, ZS =	46° 58′ 45″		
Co-latitude, PZ =	38° 28′ 10″		
2 (semi-sum, s)	152° 58′ 14″		
		log cosec	0·0121953
$(s-SZ)$	29° 30′ 22″	log cosec	0·3075794
$(s-SP)$	08° 57′ 48″	log sin	$\bar{1}$·1925739
$(s-ZP)$	38° 00′ 57″	log sin	$\bar{1}$·7894954
	76° 29′ 07″	log tan² ½SPZ	$\bar{1}$·3018440
		log tan ½SPZ	$\bar{1}$·6509220

$$\frac{\sin(s-SP)\,.\,\sin(s-ZP)}{\sin s\,.\,\sin(s-ZS)}.$$

½♌ = 24° 06′ 53·5″.

Hour angle SPZ = ♌ = 48° 13′ 47·0″.

G.A.T. = ,, ,, ,, = 3 h. 12 m. 55·13 s. = 3·2153 h.

E.T.	01 m. 51·91 s.	
3·2153 h. at 0·424 s.	01·36	
	01 m. 50·55 s.	−01 50·55

Watch correct to G.M.T. at L.M.T. 3 h. 11 m. 04·58 s.

Actually Queen Mary College is 11 s. slow of Greenwich.

Example†. An observation for time was made on Aldebaran (α *Tauri*) on Oct. 1, 1940, in latitude 52° 12′ 50″ N., the mean of two observed altitudes being 28° 36′ 20″. The average sidereal time of observing these altitudes was 0 h. 15 m. 28·4 s. by sidereal chronometer.

Find the error of the chronometer, given that the star's R.A. and declination were 4 h. 32 m. 31·1 s. and 16° 23′ 30·5″ respectively, and that the star was east of the meridian.

Co-latitude,	$ZP =$ 37° 47′ 10·0″	$(s-ZP) = 48°\ 36'\ 29{\cdot}8''$.	
Co-declination,	$SP =$ 73° 36′ 29·5″	$(s$	
Co-altitude,	$=$ 61° 23′ 40·0″	$(s$	
	2) 172° 47′ 19·5″		
Semi-sum, $s =$	23′ 39·8″	log cosec s	$= 0{\cdot}0008605$
		log sin $(s-ZP)$	$= \bar{1}{\cdot}8751809$
		log sin $(s-SP)$	$= \bar{1}{\cdot}3450079$
		log cosec $(s-SZ)$	$= 0{\cdot}3740526$
		log $\tan^2\frac{1}{2}(SPZ)$	$= \bar{1}{\cdot}5951019$
		log $\tan\frac{1}{2}(SPZ)$	$= \bar{1}{\cdot}7975510$

$$\tan\tfrac{1}{2}SPZ = \sqrt{\frac{\sin(s-ZP)\sin(s-SP)}{\sin s\,.\,\sin(s-SZ)}}$$

$= 2$ h. 8 m. 25·09 s. Hour angle is negative since the star was east.

L.S.T. $=$ 4 h. 32 m. 31·1 s. $-$ 4 h. 16 m. 50·18 s. $=$ 0 h. 15 m. 40·9 s.

Chronometer time $=$ 0 h. 15 m. 38·4 s.

Chronometer slow 02·5 s.

Wireless time signals. The systems most commonly employed are the (*a*) old international system ; (*b*) the new international system ; (*c*) the U.S.A. system ; and (*d*) the rhythmic system. The international systems are based upon the Morse letters : *O* (– – –), *N* (– ·) and *G* (– – ·) in the syllabic groups *O–NO–GO*, and the three minute intervals terminate respectively on the terminal letters of these groups, the final dot or dash in a letter being the end of a 10 sec. interval, as indicated in Fig. 124.

(*a*) In the **Old International System** the actual signal in some cases is taken at the end of the final dash of the terminal *O*, which represents an even minute, while in other cases the final dot of the *N* or the *G* is specified, giving even 10 sec. Generally any of these may be taken with sufficient accuracy for the correct signal.

h.	m.	s.	0	10	20	30	40	50	60
19	56	05							
	57	00							
	58	00							
	59	00							

FIG. 124.

This system is still used in South and West Australia, India, Ceylon, Java, Germany, and Spain.

(*b*) In the **New International System,** instead of each minute terminating in three dashes, each of 1 sec. duration, six dots are substituted at the 55, 56, ... 60 sec. of the three one-minute intervals, replacing the three dashes in Fig. 124.

This modified system has been adopted in South Africa, Victoria, France, Russia, Argentine, Brazil, and Portuguese East Africa.

(*c*) The **U.S.A. New Signal** consists of a dot at each second over a period of 5 min. with certain specified dot *omissions.* In each of the 5 min. periods, the 29th sec. together with the 56th to the 59th are invariably omitted ; in addition 51 in the first min., 52 in the second, 53 in the third, and 54 in the fourth, the entire 51 to 54 series being omitted in the fifth minute. The 60th sec., like the remainder, is indicated by a dot, except at the beginning of the fifth minute, when a 1-sec. dash is transmitted, the beginning of the dash being the time signal.

(*d*) The **Rhythmic System** is really a " time vernier " which obviates the reading of the fractional parts of a second at the minute's dash, 306 beats, equally spaced, being transmitted in 300 secs. Of these beats the dashes are given at 1, 62, 123, 184, 245, and 306, and thus 61 dots occur in a mean time period of 1 min., the beginning of each such minute being emphasised with a dash. Hence the interval between two successive dots is 60/61 sec. $=0{\cdot}9836$ sec.

Usually the observer either (i) counts the dots from each dash in turn until coincidence occurs between the beat of the signal and the tick of the chronometer ; or (ii) notes both the chronometer time of each dash and each coincidence, and by interpolation finds the exact instant at which L.M.T. coincides with signalled G.M.T.

Thus in Fig. 125 coincidence occurs at the ninth dot, or L.M.T. = 12 h. 14 m. 57 s., which corresponds with G.M.T. = 18 h. 0 m. $+9\times 60/61$ s. = 18 h. 0 m. 8·85 s.

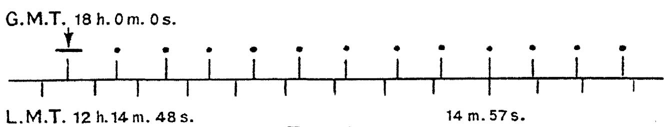


FIG. 125.

Example†. A sidereal time chronometer was believed to be fast on L.S.T. when the following data relative to the wireless time signal of 18 h. G.M.T. were recorded :

G.M.T.	18 h.	0	1	2	3	4	5	m.
L.S.T.	15 h.	42 56	43 56	44 56	45 56	46 56	47 56	m./s.
Coincidences								
	15 h.		43 04	44 04	45 04	46 05	47 05	m./s.

Shortly afterwards, at 15 h. 54 m. 24·20 s. by this chronometer, an extra meridian observation was made on a star, and the calculated hour angle was 3 h. 59 m. 15·26 s., the star's right ascension being 19 h. 53 m. 37·36 s.

Determine (*a*) the error of the chronometer on L.S.T. and (*b*) the longitude of the station, given that G.S.T. at G.M.N. was 11 h. 4 m. 27 s. (U.L.)

The seconds coincidences after the minutes dash on sidereal time are in order 8, 8, 8, 9, 9, with an average of 8·4 s., which reduced to mean time is $365/366 \times 8{\cdot}4 \times 60/61 = 8{\cdot}24$ s.

Thus G.M.T. 18 h. 0 m. 8·24 s. corresponds to L.S.T. 15 h. 43 m. 04 s. on the chronometer.

Now observed S.T. is

19 h. 53 m. 37·36 s. − 3 h. 59 m. 15·26 s. = 15 h. 54 m. 22.10 s.

(*a*) Error of chronometer on L.S.T. = 2·10 s. fast.

(*b*) Also G.S.T. at G.M.T. 18 h. 0 m. + 8·24 s. was

11 h. 4 m. 27 s. 6 h. + 6 × 9·857 s. + 0027 × 8·24 s.

	= 17 h.	5 m.	26·16 s.
Corrected L.S.T. at G.M.T. 18 h. 0 m. 8·24 s.	15	43	01·90
Long. Diff. S.T.	1	22	24·26
Long.	1	22	10·76 W.

QUESTIONS ON ARTICLE 2

1†. The following notes were recorded on June 4, 1924, in determining the local mean time, both limbs of the sun being observed.

(*a*) Latitude of place, 51° 28′ 30″ N. ; Longitude, 2° 25′ 25″ W.

(*b*) Mean G.M.T. of observations (by watch), 3 h. 1 m. 50 s.

(*c*) Mean observed altitude, 43° 2′ 10″.

(*d*) Sun's declination at G.M.N., June 4, 1924, N. 22° 26′ 6·0″ ; Variation per hour, 17·5″.

(*e*) Equation of Time, G.A.N., June 4, 1924, 1 m. 53·91 s., decreasing 0·425 s. per hour. Subtractive from apparent time.

(*f*) Correction for horizontal parallax, 8·68″.

Correction for refraction in altitude, 57″ cot α.

State the true local mean time of the observation. (U.L.)

[3 h. 11 m. 01·3 s.]

2†. At 10 h. 4 m. 2 s. a.m. Greenwich Summer Time on August 4, 1930, at a station in lat. 51° 12′ 15″ N., the altitude of the sun's centre was found to be 42° 31′.

Find the Local Mean Time and the longitude of the station.

Sun's declination at G.M.N. = 17° 22′ 35·3″ N., decreasing 39·6″ per hour.

Equation of Time at G.M.N. 6 m. 0·7 s. to be added to apparent time and decreasing 0·2 s. per hour. Refraction 62″. Parallax 7″. (U.L.)

[9 h. 16 m. 57·7 s. ; Long. E. 12 m. 55·7 s. = 3° 13′ 55″.]

3. An observation is to be made on a star *R.A.* 4 hr. 12 m. 8 s., *D.* 54° 7′ 40″. Calculate the local mean time of upper transit of the star at a place 52° 12′ N. latitude and 16° 30′ E. longitude on a day on which the G.S.T. of G.M.N. is 16 hr. 18 m. 10 s. Calculate the altitude of transit. In converting time remember that 366 sidereal days are equivalent to 356 mean solar days—a difference of 9·8 seconds per hour. (I.C.E.)

[11 h. 52 m. 11·8 s. ; 88° 04′ 20″.]

4. On a given date the L.M.T. at a place in longitude 60° W. is 8 h. 30 m. p.m. Find the L.S.T. at this instant. On the same day the G.S.T. of G.M.N. is 4 h. 42 m. 20·6 s. The acceleration of S.T. on M.T. is 9.86 seconds per hour. Work your results to the nearest tenth of a second of time. (I.C.E.)

[13 h. 14 m. 23·8 s.]

5. The Right Ascension of a star is 10 hours 23 minutes 36 seconds. Find the local mean time of transit at a place in long. 30° E. The G.S.T. of G.M.N. is found from the *Nautical Almanac* to be 2 hours 17 minutes 44 seconds. The acceleration of S.T. on M.T. is 9·86 seconds per hour, and 1 hour S.T. = 1 hour M.T. − 9·83 sec. M.T. Work results to nearest second throughout. (I.C.E.)

[8 h. 4 m. 52 s. p.m.]

6. At what hour, local mean time, at a place latitude 50° N. and longitude 60° E., was α Tauri (Aldebaran).

(*a*) at upper transit, (*b*) on the horizon in the east, on 18 May, 1940?

Given R.A. of α Tauri on 19 May was 4 h. 32 m. 29 s. and declination of the same star on the same date was + 16° 23′ 17″, also G.S.T. of G.M.T. 0 hours was 15 h. 42 m. 06·3 s. on the same date.

Also 1 hour M.T. = 1 hour S.T. + 9·8565 sec. S.T. and 1 hour S.T. = 1 hour M.T. − 9·8295 sec. M.T. (U.B.)

[(*a*) 12 h. 48 m. 55·81 s. ; (*b*) 5 h. 27 m. 47·36 s.]

7. An observation to find the error of a watch was made at Birmingham University, latitude 52° 26′ 56″ N. and longitude 07 m. 43 s. W., on 30 April, 1938.

The mean of two uncorrected vertical angles to the sun was 50° 25′ 55″. The mean G.M.T. of these two readings, as given by the watch, was 11 h. 07 m. 04 s.

Find the error of the watch on G.M.T.

Given : $\Sigma E = 16, \Sigma O = 15$, and the value of one division of the bubble = 20″.

The declination of the sun at G.M.T. 0 hours on 30 April, 1938, was 14° 29′ 59″ N. Change in declination in 24 hours = 1,110″.

G.M.T. of G.A.N. on 29 April, 1938, was 11 h. 57 m. 22·56 s.

G.M.T. of G.A.N. on 30 April, 1938, was 11 h. 57 m. 14·21 s.

Note.—Do not neglect the corrections for refraction and parallax to be applied to the mean of the vertical angles as read in the field. (U.B.)

[Watch slow 8 m. 42·6 s. on G.M.T.]

8. An observation for time was made on the sun at a place, north latitude 49° and west longitude 01°, on 15 May, 1940.

The uncorrected vertical angle of the sun's centre, the mean of two readings, was 50° 11′ 30″. The chronometer gave the average local mean time as

14 hours 01 mins. 15 sec. Find the error of the chronometer on local mean time.

Given : $\Sigma O = 32$, $\Sigma E = 24$, value of 1 bubble division $= 10''$.

Sun's horizontal parallax $= 8{\cdot}85''$.

North declination of the sun at G.M.T. 0 hours on 15 May was $18° \, 46' \, 42''$, variation in 24 hours being $848''$.

G.M.T. of G.A.N. on 15 May was 11 hours 56 mins. 15·25 sec.

Neglect the hourly variation of E. (U.B.)

[4 m. 59·3 s. slow.]

9. Write brief explanatory notes on the following :

(i) First point of Aries, (ii) equation of time, (iii) convergence of the meridians, (iv) right ascension, and (v) apparent solar time.

Prove that the equation of time vanishes four times in a year. (T.C.C.E.)

10. Show with the aid of sketches, where necessary, the relationship between the following :

(i) The R.A. of a star, the hour angle of the star at any instant, and the sidereal time at that instant.

(ii) Local mean time, local apparent time and the equation of time.

What do you understand by correction for " parallax ", " semi-diameter " and " refraction "? When are these used? (T.C.C.E.)

11. Describe briefly the order of work in the field necessary for the correction of a clock. What other data are necessary for this correction and how is it obtained? (T.C.C.E.)

12. Explain the system of time reckoning known as sidereal, apparent solar and mean solar times and show how they differ from each other. (T.C.C.E.)

13. Your longitude is 75° E. of Greenwich.

You are required to find the error to the nearest second of a mean time chronometer at midnight 1st–2nd March.

In order to do this you have timed the transit of two stars near midnight as follows :

	h.	m.	s.	
Transit of α Mali - -	23	32	14	by the chronometer
,, ,, β Gemini - -	1	43	52	

Relevant extracts from the *Nautical Almanac* are :

	h.	m.	s.
R.A. of α Mali - - -	6	19	01
,, ,, β Gemini - -	8	30	56

Sidereal time of Greenwich mean noon 1st March 18 h. 45 m. 12 s.

(T.C.C.E.)

[Chronometer slow 28 s.]

14. At Roorkee, latitude $29° \, 52'$, observations for time were taken on Oct. 17, 1937. The mean altitude obtained on α Andromeda (E) was $34° \, 11' \, 50''$ and the chronometer time was 18 h. 23 m. 10 s.

R.A. of the star from N.A. was 00 h. 04 m. 39·2 s.

S.T. at L.M.N. from N.A. was 13 h. 39 m. 07·2 s.
Declination of the star from N.A. was 28° 41′ 34″.
Find the chronometer error. (T.C.C.E.)
[18 m. 15·5 s. fast.]

ARTICLE 3 : AZIMUTH

An **azimuth** is the horizontal angle a celestial body makes with the pole, and thus conversely establishes the direction of the true meridian with respect to a **referring object**, which often is a station of the survey, appropriately illuminated as a backsight in night observations.

Four general methods will be considered in the assumed scope of the subject.

First method. (*a*) **By equal altitudes of a circumpolar star.** Since circumpolar stars appear to circle around the pole, it is evident that a given star s_1 observed at an altitude α east of the pole will after a certain interval of time appear at s_2 at the same altitude α west of the pole; and if the corresponding positions are fixed by angles θ_1 and θ_2 from an illuminated backsight or object, the meridian will bisect the angle $(\theta_1 - \theta_2)$, or, in other words, the line AB will make an angle $\theta = \frac{1}{2}(\theta_1 + \theta_2)$ with the meridian; that is, will have a bearing of N θ W or $360° - \theta$, as in Fig. 126.

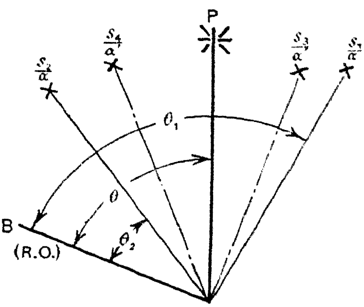


FIG. 126.

This is the simplest method of determining azimuth, the observation being independent of latitude, longitude, and time, and involving no calculation beyond the addition or subtraction of horizontal angles. On the other hand, the duration of the work is a great inconvenience, extending from four to six hours at night. Also the effects of atmospheric refraction may vary considerably during the interval, affecting the vertical angles to an unknown extent. The risk of the star being overcast at the instant of the second observation is obviated by observing a second altitude α' for the positions s_3 and s_4, or even more; and if these are satisfactorily paired, averaging out the horizontal angles, the method can be comparatively precise. Obviously the star may be observed on its lower path if desired.

(*b*) By **equal altitudes of the sun.** The corresponding method of observing the sun is academic rather than practical; and a correction must be applied to the mean horizontal angle β between the two observed positions of the sun to correct for the change in its declination during the interval between the observations; for when the declination is altering towards the north (or south) the approximation to direction of the meridian, as determined by the mean of the sun's observed positions, is too far to the right (or left), and the consequent correction is equal to $\frac{\frac{1}{2}d\delta \sec \lambda}{\sin \frac{1}{2}\beta}$, where λ is the latitude, and $d\delta$ the change in the sun's declination in the interval.

Second method. (*a*) **By a circumpolar star at elongation.** This method possesses the advantages of methods involving one observation of one star, and results to within 15″ can be obtained with a six-inch transit theodolite. Polaris is the star most commonly observed in the northern hemisphere. Since this star does not move east or west more than 5″ in a period of ten minutes before and after elongation, ample time is at the surveyor's disposal for his two observations, the first with the telescope normal and the second with it inverted. In America, tables are developed giving the azimuths of the star for a considerable number of years and a wide range of latitudes, corresponding tables of times of elongation being also available . A table of pole star azimuths is given in the " Abridged Nautical Almanac ".

Of the quantities involved in the solution of the triangle *SPZ*, the declination offers no difficulty, while the latitude if not otherwise known may be accounted, or at worst scaled, from a map ; but the hour angle involves a knowledge of sidereal time unless complete tables are available, in order that the time of elongation may be found.

A star is at **eastern elongation** (E.E.) when it is exactly east of the pole *P*, and at **western elongation** (W.E.) when it is exactly west of the pole, the respective intervals before or after the star's upper culmination, or transit, being from 5 h. 45 m. to 5 h. 54 m. of mean time, according to the latitude.

The calculations are based upon the fact that, in the spherical triangle *SPZ*, the parallactic angle *PSZ* is 90°, a fact that will be evident on drawing in plan the tangents from a station *Z* on the meridian to the star in its circular path about a centre at the pole *P*.

$$\text{(i)}\ \sin SZP = \sin A = \frac{\cos\delta}{\cos\lambda};\quad \text{(ii)}\ \cos PSZ = \cos \Omega = \frac{\tan\lambda}{\tan\delta};$$

$$\text{(iii)}\ \cos\alpha = \frac{\sin\lambda}{\sin\delta}.$$

The interval in sidereal time before or after culmination is found from (ii), which is subtracted from or added to the time of culmination, and this

is reduced to a mean time equivalent (p. 278). At this instant the star is observed, and the horizontal angle from the backsight duly recorded, this measurement being the sole field work involved. The azimuth is then computed by means of (i).

The accuracy with which the latitude must be known can be found by differentiating (i), since only the date of the declination need be known :

$$dA = \tan A \tan \lambda \, . \, d\lambda,$$

which indicates that the error dA in azimuth increases both with the distance of the star from the pole and with the latitude, the error in the assumed value of which is $d\lambda$.

An error of $1'$ in latitude (which is more than 1 statute mile) will introduce an error of only about $1''$ or $2''$ in the calculated azimuth, while the declination, as obtained from the date and Nautical Almanac, will be correct to about $1''$.

(*b*) By **two circumpolar stars at elongation.** The latitude λ and the error in its assumed value $\delta\lambda$ will be eliminated when observations are made on two stars, which are either nearly in opposition or conjunction, and will therefore elongate at nearly the same time. The angles θ_1 and θ_2 to be measured are those between the backsight and each of the stars at elongation, the backsight often being the line whose bearing is to be determined. Then the *difference* or *sum*, $\theta_1 \mp \theta_2$, of these angles is the angle β between the stars according as the backsight is to one side of the stars or is between them. This angle β is obviously the sum or difference of the star's azimuths, $A_1 \pm A_2$, according as these are respectively measured at opposite or the same elongation. It is always advisable to work thus from first principles rather than to confuse signs in the final expression.

$$\sin A_1 = \frac{\cos \delta_1}{\cos \lambda}\,; \quad \sin A_2 = \frac{\cos \delta_2}{\cos \lambda}\,; \quad \frac{\sin A_1}{\sin A_2} = k = \sin (\beta \mp A_2),$$

since

Hence $$k = \sin \beta \cot A_2 \mp \cos \beta,$$

and $$\tan A_2 = \frac{\sin \beta}{k \pm \cos \beta},$$

the $\genfrac{}{}{0pt}{}{plus}{minus}$ signs indicating that β is $A_1 + A_2$

As in the preceding method, the times of elongation of the stars must be determined.

Third method. (*a*) By **circumpolar stars in the same vertical.** This method consists in tabulating the data relative to pairs or stars s_1 and s_2, such as β and ϵ Ursae Majoris and β and δ Draconis, so as to observe these in the same vertical ZZ'.

Sine of azimuth $PZs_1 = \sin Ps_1 . \sin Ps_1s_2 . \sec \lambda$, which is reduced to $\sin A = k \sec \lambda$, since the product of the angle and side is constant to within a few seconds throughout the year.

(*b*) By the **time intervals of two circumpolar stars.** The following is a delicate test as to the accuracy of a meridian already determined rather than a method of correcting an approximate meridian.

Two stars, such as β Cassiopeia and γ Ursae Majoris, which differ in right ascension by nearly twelve hours, are selected, and the exact times of their opposite culminations are computed. The instrument is levelled up with extreme care, and the telescope is clamped with its cross-hairs exactly bisecting a point indicating the north of the meridian already determined. The exact times of opposite culmination are observed. Then, if the observed meridian is also a true meridian, the computed sidereal interval will have elapsed between the culminations. But if the observed meridian is to one side, the observed time interval will be greater or less; and if the rate of travel in azimuth of the stars be known, it is possible by simple proportion to eliminate the error.

As in the preceding method, the accuracy of this test depends upon the precision of the instrument in vertical adjustment.

Fourth method. (*a*) **By one extra meridian observation of the sun.** Doubtless this is the surveyor's most convenient method of determining a meridian, and hence its application through the medium of the solar attachment. Only one observation is involved, and that may be made at any convenient time. The field work is of but a few minutes duration; the calculations, half an hour at most. As in the case of extra-meridian observations for time, the three sides are known, but the angle required in solar azimuth is SZP. Either the sine or tangent formulae may be used, preferably the latter:

$$\tan \tfrac{1}{2}SZP = \sqrt{\frac{\sin(s - SZ) . \sin(s - PZ)}{\sin s . \sin(s - SP)}},$$

where s is the semi-sum of the sides, SZ, SP, and ZP.

Since logarithmic reduction is usually followed, the foregoing may be written:

$$2 \log \tan \tfrac{1}{2}(SZP)$$
$$= \log \sin(s - SZ) + \log \sin(s - PZ) + \log \operatorname{cosec} s + \log \operatorname{cosec}(s - SP).$$

These formulae give the angle as measured from the *north* point in a *clockwise* or *counter-clockwise* direction, according as the sun is *rising* or *setting*. Expressed as a clockwise whole circle bearing from the *north*, the azimuth in the former case is merely the angle PZS, and in the latter case $360° - SZP$.

As in all observations for azimuth, the horizontal angles from a back-sight or referring point are read and recorded, the mean being taken when

opposite limbs are observed in order to eliminate instrumental defects and to avoid the correction for semi-diameter, which otherwise would have to be taken into account. Solar observations should never be made within an hour of noon ; 8 to 10 in the forenoon and 2 to 4 in the afternoon being the best periods.

The accuracy with which the altitude is to be observed and with which the latitude and declination are to be reduced, depends upon the objects of the work. The latitude may be scaled from a map, found by account from a previous station, or determined by independent observation, these methods suggesting three distinct degrees of accuracy. The degrees of approximation to the declination have been discussed (p. 272). The respective errors in azimuth due to errors in altitude ($d\alpha$), in latitude ($d\lambda$), and in declination ($d\delta$), will follow by differentiation of the formula

$$\cos SZP = \cos A = \frac{\sin\delta - \sin\lambda\sin\alpha}{\cos\lambda \,.\, \cos\alpha};$$

$$dA_1 = +\frac{d\alpha}{\tan PSZ \,.\, \cos\alpha}, \quad dA_2 = -\frac{d\lambda}{\tan \Omega \,.\, \cos\lambda}; \quad dA_3 = -\frac{d\delta}{\sin \Omega \,.\, \cos\lambda},$$

Ω being the angle SPZ and PSZ the parallactic angle.

(*b*) **By an extra-meridian observation of a star.** In theory this and the preceding method are identical. S here denotes a star which is a point at an infinite distance, and hence semi-diameter and parallax are not to be considered. Also the time of the observation is relatively unimportant except in so far as it affects the hour angle. In application, therefore, the stellar method is simpler, but obviously less convenient, apart from the difficulties of illuminating the axis and reference point.

(*c*) **By mechanical solution with the solar attachment.** The mechanical solution of the astronomical triangle is exceedingly convenient in observations where great accuracy is unnecessary, as in extensive preliminary and route surveys. Remarkably accurate results can be obtained with the instrument when used with skill and understanding.

The best-known patterns of solar attachments are those of Burt and Saegmuller (p. 88), both of which are fitted to the theodolite. A time for the proposed observation is fixed tentatively, preferably between 8 and 10 in the forenoon and from 2 to 4 in the afternoon, though actually four or five different times, varying by 10 minutes, are selected, primarily against the contingency of the sun being obscured at any of these. For each selected time the apparent declination of the sun at Greenwich reduced with due regard to local mean time and longitude, or standard time alone if this is kept in the observations. These declinations are corrected for refraction in declination, usually from special tables, as those in the author's " Field Manual " (see p. 267). The corrected value for the first of the proposed times is set off on the declination arc of Burt's

attachment, the arc being turned towards the eyepiece when the declination is north (+) and towards the objective when the declination is south (−). Also the vertical circle is clamped at an angle of depression equal to the co-latitude of the place. In the case of the Saegmuller attachment, however, the declination is set off on the vertical circle with an elevated telescope for *plus* values and a depressed telescope for *negative* values. The solar telescope is then levelled. Next the vertical circle is unclamped and set at an angle of elevation equal to the co-latitude of the place. Finally, in either case the telescope is turned about the lower motion while the solar telescope or the arc is turned about its polar axis until the sun's image appears in the square of the image plate, or at the centre of the cross-hairs, at the time for which the declinations are computed. The line of sight of the telescope will point due south in the meridian after a fine adjustment with the lower tangent screw of the theodolite, turning also (if necessary) the arc or telescope of the attachment.

Illustrative Examples

OBSERVATION for Azimuth......BY......Polaris at Western Elongation.
PLACE......East London College. DATE......January 9/10, 1923.
TIME......G.M.T. BY......Watch.
LATITUDE......N. 51° 31′ 50″. LONGITUDE......0° 02′ 45″ W.
REFRACTION...........BAROMETER...........THERMOMETER...........
REFERRING OBJECT......Illuminated slit, W. of star.
INSTRUMENT......Cooke Micrometer, T 1 ; OBSERVER ....
STATION......
DECLINATION...88° 53′ 50·64″ N. R.A....1 h. 33 m. 50·20 s.
G.S.T. (G.M.N.)......19 h. 11 m. 59·8 s.

Object	Face of instr.	Horizontal angle		Vertical angle		Level		Time
		A	B	C	D	$d=15''$		
		Micr.	Micr.	Micr.	Micr.	O	E	h. m. s.
R.O.	R.	104° 06′ 00″	220° 06′ 00″			4·0	3·8	12 02 00
STAR	R.	122° 30′ 10″	302° 30′ 10″					12 12 00
STAR	L.	302° 30′ 20″	122° 30′ 20″			3·8	4·0	12 17 00
R.O.	L.	284° 06′ 10″	104° 06′ 10″					12 24 00

(Remainder of form detached)

		Reducing this arc ♌ to sidereal time.			
Log cot 88° 53′ 50·64″	2·2843425	88° × 4 m.	5 h.	52 m.	00 s.
Log tan 51° 31′ 50·00″	0·0998703	37′ × 4 s.		02	28·00
Log cos ♌	2·3842128	1/15(16·74) s.			01·12
♌ = 88° 37′ 16·74″		♌ =	5	54	29·12
		R.A.	1	33	50·20
		R.A. + ♌ = L.S.T. of Elongation	7	28	19·32
G.S.T. (G.M.N.)	19 h. 11 m. 59·80 s.	Add	24	00	00·00
Long. W. 2′ 45″ :	11 s. × 0·003 = 0·03		31	28	19·32
L.S.T. (L.M.N.)	19 11 59·83		19	11	59·83
		S.I. after L.M.N.	12	16	19·49
Reducing to mean time (1 m. 57·96 s. + 2·62 s. + 0·05 s.)				02	00·63
		L.M.T. of W. Elongation	12	14	18·86
		Add for watch time			11·00
		Mean sight at 12.15 a.m., Jan. 10, G.M.T.	12	14	29·86

$$\text{Sin } SZP = \sin A = \frac{\cos \delta}{\cos \lambda}.$$

Log cos 88° 53′ 50·64″ = $\bar{2}$·2842622
Log cos 51° 31′ 50·00″ = $\bar{1}$·7938582

Log sin A = $\bar{2}$·4904040 : A = 1° 46′ 21·21″
Mean observed angle between R.O. and star = 18° 24′ 10·00″

True bearing of R.O. N. 20° 10′ 31·21″ W.
Azimuth of R.O. 339° 49′ 28·79″

OBSERVATION for Azimuth......BY......Extra Meridian Observation of Sun.
PLACE......Quex Park, Birchington. DATE......September 24, 1928.
TIME......G.M.T. BY......Watch (Summer Time).
LATITUDE......N. 51° 21′ 57″. LONGITUDE......
REFRACTION 57″ cot α.......BAROMETER......THERMOMETER. ......
REFERRING OBJECT OR BACKSIGHT......Sta. P for Azimuth of Base OP.
INSTRUMENT......Cooke Micrometer, T_1; OBSERVERS......B. F. S. & B. S. J.
STATION......O, on base of triangulation.
DECLINATION.....G.M.N. Sept. 24, S. 0° 28′ 08″, increasing south at 0·97′/hr.
RIGHT-ASCENSION...........
G.S.T. (G.M.N.).............. EQUATION OF TIME...........

Object	Face of instr.	Horizontal angle		Vertical angle		Level		Time		
		A	B	C	D	$d = 15''$				
		Micr.	Micr.	Micr.	Micr.	O	E	h.	m.	s.
	L	311° 41′ 15″	311° 41′ 47″	23° 40′ 38″	23° 40′ 38″	4·5	4·5	15	07	18·0
	R	313° 56′ 40″	313° 56′ 04″	22° 22′ 36″	22° 22′ 34″	4·0	5·0	15	13	15·4
						8·5	9·5	15	10	14·7
Mean Horizontal Angle, 312° 48′ 56·5″ ; Mean Vertical Angle, 23° 01 36·5″										

SEMI-DIAMETER........... MEAN OBSERVED ALT. 23° 01′ 36·5″

INDEX ERROR..............

LEVEL CORRECTION $\frac{1}{4}(8{\cdot}5 - 9{\cdot}5)\,15'' = -03{\cdot}8''$

(1) NET INSTRUMENTAL CORRECTION −03·8″ 03·8″

MEAN APPARENT ALT. 23° 01′ 32·7″

REFRACTION CORRECTION : 57″ cot (23° 1·5′)

= 02′ 14·1″

PARALLAX : 8·77″ cos (23° 1·5′) 08·1″

(2) NET OBSERVATIONAL CORRECTION −02′ 06·0″ −02′ 06·0″

TRUE ALTITUDE 22° 59′ 26·7″

Co-declination SP = 90° 31′ 12·5″ Declination (G.M.N.) S. 0° 28′ 08·0

Co-altitude SZ = 67° 00′ 33·3″ Add 3·1707 × 0·97′ 03′ 04·5

Co-latitude PZ = 38° 38′ 03·0″ 0° 31′ 12·5

2s) 196° 09′ 48·8″

Semi-sum s	98° 04′ 54·4″	Log cosec	0·0043348
$(s - SP)$	07° 33′ 41·9″	Log cosec	0·8807694
$(s - SZ)$	31° 04′ 21·1″	Log sin	1·7127531
$(s - PZ)$	59° 26′ 51·4″	Log sin	1·9356614
CHECK	196° 09′ 48·8″	2 Log tan $\frac{1}{2}SZP$)	·5335187

$\frac{1}{2}SZP$ = 61° 35′ 03·16″ Log tan $\frac{1}{2}SZP$ 0·2667593

SZP = 123° 10′ 06·3″ ; or 236° 49′ 53·7″ clockwise from north.

Azimuth of OP = (360° + 236° 49′ 53·7″) − 312° 48′ 56·5″

= 284° 00′ 57·2″.

N.B.—Declination (G.M.N.) taken 4″ too small ; hence error in azimuth is cosec (47° 34′) sec (51° 22′4″) = 8·7″. (See p. 297.)

Example†. Criticise the method of determining azimuths from elongation observations, stating its limitations in high latitudes.

A star α of Declination 84° 42′ N. is observed at eastern elongation when its clockwise angle from a survey line is 118° 20′. Immediately afterwards another star β of Declination 72° 24′ N. is observed at western elongation, its clockwise angle from OP being 94° 6′.

Determine the azimuth of the line OP. (U.L.)

Let A_1, A_2 be the azimuths of the stars α and β at opposite elongations, δ_1 and δ_2 their respective declinations and β the horizontal angle between them, the line OP lying to the left of the stars.

$$\frac{\quad}{\sin A_2}=\frac{\quad-\delta_1)}{\sin(90^\circ-\delta_2)}=\frac{\sin 5^\circ\,18'}{\sin 17^\circ\,36'}=k=0{\cdot}30549=\frac{\sin(\beta-A_2)}{\sin A_2},$$

$$\tan A_2=\frac{\sin\beta}{k+\cos\beta}=\frac{\sin 24^\circ\,14'}{k+\cos 24^\circ\,14'}=0{\cdot}337164\,;\quad A_2=18^\circ\,37'\,56''.$$

Azimuth of $OP=94^\circ\,6'+18^\circ\,37'\,56''=112^\circ\,43'\,56''$, or $247^\circ\,16'\,04''$.

Example. A star of declination 17° 32′ 10″ N. is observed before upper transit in lat. 50° 36′ 25″ N. at a true altitude of 40° 25′ 40″, when its clockwise bearing from a survey line is 125° 30′ 15″.

Determine the azimuth of the line clockwise from the north. (U.L.)

$$\tan\tfrac{1}{2}SZP=\sqrt{\frac{\sin(s-SZ)\,.\,\sin(s-PZ)}{\sin s\,.\,\sin(s-PS)}}=\tan\,.\,\tfrac{1}{2}(57^\circ\,13'\,06''),$$

or $SZP=114^\circ\,26'\,12''$ eastwards from north pole, since star is rising.

Azimuth of line : $360^\circ-(125^\circ\,30'\,15''-114^\circ\,26'\,12'')=348^\circ\,55'\,57''$.

QUESTIONS ON ARTICLE 3

1†. The following observation on a star at western elongation was made at a station O in lat. 52° 20′ N. and long. 52° 20′ E., the referring object being another station P in the survey.

Object	Horizontal angle		Data
	A. Micr.	B. Micr.	
P	64° 5′ 25″	244° 5′ 15″	Decl. 74° 27′ 30″ N.
Star	169° 55′ 20″	349° 55′ 10″	R.A. 14 h. 50 m. 54 s.
			G.S.T. (G.M.N.) 5 h. 16 m. 54 s.

Determine (a) the azimuth of the line OP ; (b) the L.M.T. of the observation. [228° 09′ 40·6″ ; 2 h. 7 m. 47·1 s.] (U.L.)

2†. A star was observed at western elongation at a place in lat. 28° 20′ S. and long. 124° 24′ W., when its clockwise bearing from a survey line was 164°.

Determine the local mean time of elongation, also the azimuth of the line, given that the star's declination was 76° 36′ 55″ S., and its right ascension 6 h. 41 m. 52 s., the G.S.T. of G.M.N. being 5 h. 12 m. 20 s. (U.L.)

[7 h. 58 m. 19·13 s., 180° 45′ 7·75″ from S. point]

3†. The following notes were recorded on May 13, 1920, in determining the azimuth of a reference point from Station 780 of a railway survey, the sun's lower limb being observed :

(*a*) Latitude of Station 780, 53° 30′ N.

(*b*) Mean of G.M.T.'s of two observations, 10 h. 20 m. a.m.

(*c*) Mean observed altitude, 50° 24′ 40″.

(*d*) Mean observed horizontal angle of sun, right of reference point, 42° 12′.

(*e*) Sun's declination at Greenwich Mean Noon, May 12, 1920, N. 18° 7′ 38·0″. Variation per hour, + 37·79″.

(*f*) Correction for horizontal parallax, 8·71″. Correction for semi-diameter, 15′ 51·10″. Correction for refraction, 57″ cot α, where α is the mean apparent altitude.

State the azimuth of the reference point. (U.L.)

[98° 55′ 3″]

4†. The following notes were recorded on September 21, 1931, in finding the azimuth of a reference point from station *Q* of a triangulation survey, the upper and lower limbs of the sun being observed in using opposite faces of the theodolite :

(*a*) Latitude of station *Q*, 51° 22′ 12·3″ N.

(*b*) G.M.T. of the two observations, 14 h. 11 m. and 14 h. 18 m.

(*c*) Mean apparent altitude, 30° 58′ 43·9″.

(*d*) Mean horizontal angle of sun, to right of reference point, 62° 7′ 14·5″.

(*e*) Declination at G.M.N., N. 0° 58·8′ decreasing 0·97′ per hour.

(*f*) Correction for horizontal parallax, 8·78″.

(*g*) Correction for refraction, 57″ cot (apparent altitude).

Determine the azimuth of the reference point. (U.L.)

[161° 50′ 39·5″]

5†. Observations for azimuth are to be made with a Saegmuller Solar Attachment near Montevideo, Uruguay, in latitude by account 34° 55′ 20″ S., on November 9, 1940, at selected intervals beginning at 15 h. 00 m. 00 s. by the mean time of the 60th West meridan.

Describe in detail how you would carry out the observations with the aid of tables of refraction in declination, given that the sun's declination at G.M.M. on date is 16° 45·5′ S., increasing 0·70′ per hour.

6. At a place of which the latitude and longitude are known the altitude of a star and its bearing from a mark are observed. The Greenwich Mean Time of the observation is noted.

Explain how, with the help of the *Nautical Almanac*, the azimuth of the mark and the error of the clock can be computed.

The explanation must be full and illustrated with a diagram of the celestial sphere on which the various observed or calculated angles are shown. (I.C.E.)

7. An observation for azimuth was made during the early hours of the morning of 1 January, 1940, on α Ursae Minoris (Polaris) at elongation at a place latitude 45° N. and longitude 5° E.

The declination of the star on that date was $+88° 59' 03''$ and its R.A. was 1 hr. 43 m. 32 s.

The mean observed horizontal angle between the star and the R.O. was $42° 37' 22''$, the R.O. being to the west of the star.

Find (*a*) which elongation was used,

(*b*) the exact local mean time of this elongation,

(*c*) the azimuth of the R.O.

Given G.S.T. of G.M.T. 0 hours on 1 January, 1940, was 6 h. 38 m. 01·9 s. (U.B.)

[(*a*) West; (*b*) 0 h. 57 m. 27·6 s. Jan. 2; $315° 56' 38''$.]

8. State Napier's rules of circular parts for the solution of right angled spherical triangles.

"z" Draconis "w" was observed in latitude $29° - 52'$ and longitude $77° - 54'$ on the evening of November 16, 1931. The following data was obtained from the *N.A.*:

the star's declination was - - -	N. $65° - 47' - 54{\cdot}5''$
the star's R.A. was - - - -	17 h. – 08 m. – 32·3 s.
and the sidereal time of L.M.N. was -	15 h. – 37 m. – 32 s.

Find the chronometer time of elongation and its altitude and azimuth at that instant. (T.C.C.E.)

[18 h. 30 m. 6·5 s. L.M.T. (W.E.); $33° 5' 27{\cdot}2''$; $28° 12' 42{\cdot}5''$.]

9. In what part of the sky would you select stars for the observation of time and azimuth?

Describe briefly how you would make an accurate observation of azimuth in the field using a theodolite. Mention the order of work and state the formula you would employ for the determination of azimuth after the field observations. (T.C.C.E.)

10. State the different approximate and correct methods by which the azimuth of a survey line can be found astronomically. (T.C.C.E.)

11. The following readings were made at Nowshera, h.s., latitude $34° 01' 47{\cdot}1''$.

	Horizontal arc	Vertical arc
Jalala Sar, H.S.	$204° 02' 19''$	
Polaris	$357° 14' 21''$	$34° 56' 02''$

What is the astronomical azimuth of Jalala Sar, H.S., at Nowshera, h.s.?

Extract from the *Abridged Nautical Almanac*:

	Right ascension	Declination
Polaris	01 h. 41 m. 56 s.	N. $88° 58{\cdot}8'$

(T.C.C.E.)

[$206° 10' 37{\cdot}4''$ clockwise from N. with star W. of N.]

ARTICLE 4 : LATITUDE

There are *two general methods* of determining the latitude of a station in field astronomy. The former is based upon the fact that the altitude of the pole is equal to the latitude of the place of observation. Wherefore if a star were exactly coincident with the pole, the latitude could be directly ascertained.

First method. (*a*) **By meridian altitudes of a star.** Since the polar distance is the same at the upper and lower culminations of a circum-polar star, the mean of the corresponding true altitudes is the altitude of the pole and also the latitude. This fact introduces an exceedingly exact but lengthy method of determining the latitude ; an absolute method, since the direction of the meridian is established in observing the altitude of the star ; and neither the declination nor the time need be known. It is, however, a method to be resorted to only when the data obviated are not available ; for, in general, one of the culminations will occur in day-light if successive culminations are observed. Hence, it is usually superseded by the following method, which is shown in Fig. 127 ; and in this connection the figure should be drawn in any particular case, and the memorising of formulae avoided.

(*b*) **By the meridian altitude of a star.** In Fig. 127, O represents the earth's centre ; Z, the observer's zenith ; P, the celestial north pole ; S, a star at its upper culmination ; and HH, the true horizon. The angle HOS is the observed altitude α, and the angle $SOZ = 90° - \alpha$ is the zenith distance ZS. Also the angle POH is the angle SOH − the angle SOP ; that is :

Altitude of pole = latitude = altitude − co-declination,

or $$\lambda = \alpha - (90° - \delta). \qquad (1)$$

The case of a star S_2 in the southern hemisphere leads to the same expression.

When the star is on the other side of the zenith, as at S_1, it is said to culminate to the south, in which case the foregoing is replaced by the following expression :

Angle ZOP = angle POS_1 − angle ZOS_1 ; that is,

co-latitude = co-declination − co-altitude ;

or $$(90° - \lambda) = (90° - \delta) - (90° - \alpha) = \alpha - \delta. \qquad (2)$$

The case of the star S_3 in the southern hemisphere leads to the same expression.

The eight cases that may arise are indicated in their respective sectors in Fig. 127. The corresponding formulae may be expressed by two general rules, namely :

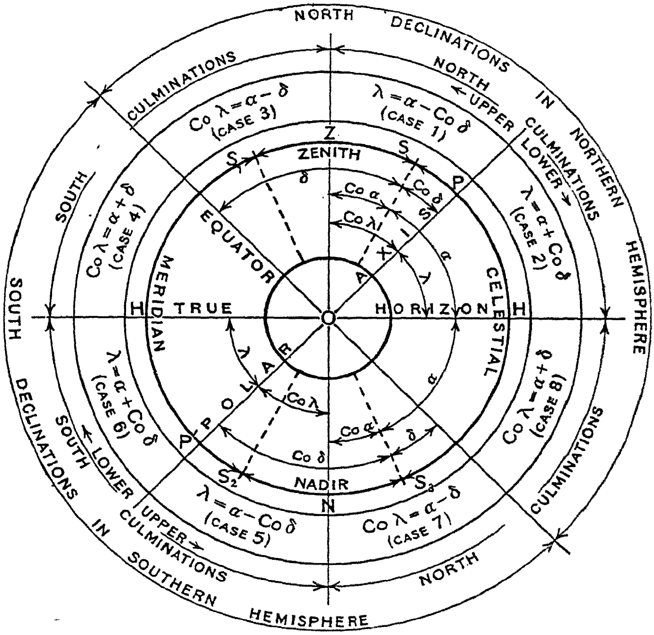


FIG. 127.

(1) When the direction of culmination and hemisphere of observation are denoted by the *same* letter (N or S), use

regarding the relative positions of the $\mp$ signs as indicating $\begin{cases}\text{upper}\\ \text{lower}\end{cases}$ culminations.

(2) When the direction of culmination and hemisphere of observation are denoted by *opposite* letters (N and S), use

regarding the relative positions of the $\mp$ signs as indicating declinations of $\begin{cases}\text{the same}\\ \text{opposite}\end{cases}$ letter $\begin{cases}\text{as}\\ \text{to}\end{cases}$ that of the hemisphere of observation.

The field work consists merely in observing the apparent altitude of culmination of a star. Being the meridian altitude, this angle is to be observed when the star crosses the *known* meridian, or at the computed local mean *time* of culmination. It is not, however, necessary that the altitude should be measured precisely at culmination, the motion of a circumpolar star being sensibly horizontal for about 15 minutes at cul-

mination. Thus it is possible to use both faces of the theodolite in the observation.

The apparent altitude needs correcting *only* for refraction, parallax being a negligible consideration. Strictly, the correction for refraction varies with the temperature and pressure of the atmosphere and, to be precise, the readings of the thermometer and barometer should be taken at the time of observation. The refraction differs at most by only 3″ per 10° F. and 5″ per inch of barometer from the values for a mean temperature of 50° F. and a mean pressure of 30 inches, the conditions for which refraction tables are usually computed. Hence temperature and pressure are negligible in much ordinary work, particularly with vernier reading instruments, with which, incidentally, the declinations are taken frequently to the tabular value for January 1. For exceedingly high elevations with very high temperatures, the variations in refraction should be considered in regard to their effect upon the accuracy of observations.

(*c*) **By the meridian altitude of the sun.** All things considered, the sun is the most convenient celestial body to observe. The method of single meridian altitudes is certainly rendered more complex; its results, less accurate; but, on the other hand, the observations are made in daylight, and the sun is an unmistakable object to anyone unfamiliar with astronomy.

The fact that the sun's hourly change in declination is appreciable involves longitude and local mean time, while semi-diameter and parallax are corrections additional to refraction in ascertaining the true meridian altitude. Nevertheless, it is the navigator's method of finding his latitude. He uses the sextant, and observes the altitude above the visible horizon, and applies an additional correction for the *dip* of the horizon in ascertaining the true meridian altitude.

The eight cases of observation (as arise in the preceding method) are reduced to four, since, in the meridian, the sun's culmination is invariably to the $\left\{\begin{matrix}\text{south}\\ \text{north}\end{matrix}\right.$ in the $\left\{\begin{matrix}\text{northern}\\ \text{southern}\end{matrix}\right.$ hemisphere. Thus only equation (2) is to be considered:

$$\text{Co} . \lambda$$

the relative positions of the $\mp$ signs indicating declinations of the $\left\{\begin{matrix}\text{same}\\ \text{opposite}\end{matrix}\right.$ letter $\left\{\begin{matrix}\text{as}\\ \text{to}\end{matrix}\right.$ that of the hemisphere of observation. For any time of the year (except at the equinoxes) one sign applies to one hemisphere; the other sign to the other hemisphere.

(*d*) **By zenith-pair observations of stars.** A zenith-pair observation is a repetition of (*b*) in which the selected stars transit within half an hour of each other at nearly equal altitudes but on opposite sides of the zenith. Actually four cases occur, according as one star S_1 is at upper or lower transit and the right ascensions are nearly the same or differ by nearly twelve hours.

Every case should be deduced from the simple relations for each star, and formulae should not be committed to memory. Each altitude should be corrected for refraction, though, since the altitudes will be nearly equal, the effects will be almost the same, and any discrepancy in the assumed refraction effect will be practically eliminated in the difference of the altitudes, which is a term common to the four cases. Even if the face of the theodolite is not changed (remaining on the same side of the observer), the index error of the vertical circle, though the same for both altitudes, will be practically eliminated in the small difference of altitude involved. Otherwise, one observation face left and the other face right, will eliminate the difference, and this procedure is favoured by most surveyors. Hence, face reversals on each star are not to be considered, since no horizontal angle is measured. In either case it is necessary to correct each altitude for bubble error in accordance with $\frac{1}{2}(\Sigma O - \Sigma E)d''$, where O and E are the bubble readings at the objective and eyepiece ends of the altitude level on the clipping arm, the value of its bubble divisions being d''.

(*e*) By **circum-meridian altitudes of a star.** Instead of observing the altitude in meridian of one or a pair of stars, greater accuracy can be obtained by taking a series of altitudes, four or five (say) *before* and *after* transit with appropriately changed faces of the theodolite, the entire series of observations not extending over more than 20 minutes. Each of the observations is corrected to give the maximum (or minimum) altitude, and the average of the values thus "reduced to meridian" is used in calculating the latitude from the relations of a single altitude (p. 304).

The reduction formula is $\alpha_0 = \alpha + mC$, with $C = \dfrac{\quad}{\cos \alpha}$ where α is the relevant altitude and λ the latitude approximated to from the results, while $m = \dfrac{2 \sin^2 \frac{1}{2}\omega}{\sin 1''} = \text{vers}\, \omega \,.\, \text{cosec}\, 1''$, ω being the hour angle in seconds of arc.

Alternative forms of m are given, frequently with ω replaced by t in minutes or seconds of sidereal time. Thus, $m = 0{\cdot}0005454t^2$ and $m = 1{\cdot}963t^2$ with t in seconds and minutes respectively.

(*f*) By **circum-meridian altitudes of the sun.** The foregoing method may be applied to solar observations when circumstances preclude working at night. In this case it is usual to sight the upper and lower limbs of the sun alternately, letting the vertical hair of the diaphragm bisect the image in order to eliminate semi-diameter horizontally. Wherefore pairs of observations cannot be used as in the case of stars, and, in consequence, uncertainty as to refraction exists, while a correction for parallax must be applied. Also the value of the declination at the instant of transit must be known; though, on the other hand, the time intervals will not have to be reduced to sidereal minutes or seconds.

Second method. (*a*) **By one extra-meridian observation of the sun.** Although the latitude may be determined at any time by solving the solar astronomical triangle SPZ, the method is not popular on account of the fact that in addition to the known sides : (1) the co-altitude SZ, and (2) the co-declination SP, (3) the sun's azimuth SPZ must be known, together with a fourth quantity determinate from these, namely, (4) the hour angle SPZ, which exists in the relation

$$\sin SPZ = \sin SZ \frac{\sin SZP}{\sin SP}.$$

Then

$$\tan \tfrac{1}{2} PZ = \frac{\sin \frac{1}{2}(SZP + SPZ)}{\sin \frac{1}{2}(SZP - SPZ)} \tan \tfrac{1}{2}(SP - SZ)$$
$$= \tan \tfrac{1}{2}(90 - \lambda).$$

When the direction of the meridian is not known, the hour angle SPZ must be deduced from the local mean time as observed with the chronometer. The azimuth SZP can then be found and its value inserted in the second equation. If SZP is less than SPZ, merely interchange these and SP and SZ.

(*b*) **By an extra-meridian observation of a star.** In theory, this and the preceding method are identical, S here denoting a star, the distance of which precludes the correction for parallax in altitude observations. Likewise variations in declination are not to be considered ; but the hour angle will be expressed in sidereal time, reduced to arc.

(*c*) **By mechanical solution with the solar attachment.** It is not possible to solve the present general problem by means of the solar attachment, and the scope of the instrument is reduced to cases in which either (i) the direction of the meridian is known ; or (ii) the sun is at transit.

Here, as in azimuth observations, the sun's declination reduced to the Greenwich meridian for local mean time and longitude is corrected for refraction in declination for a series of times (or for apparent noon only) ; and these values are set off on the declination arc of the Burt attachment or on the vertical circles of theodolites fitted with Saegmuller's attachment (p. 297).

(i) The theodolite is levelled up in the meridian with the cross-hairs exactly on the north point with Burt's attachment and on the south point with Saegmuller's attachment. The arc of the former device is turned towards the eyepiece or objective according as the declinations are *plus* or *minus*, and the vertical circle is clamped at an angle of *depression* equal to the estimated co-latitude of the place. After the corrected value of the declination has been set off on the vertical circle, with a depressed or elevated telescope, according as the declination is *plus* or *minus*, Saegmuller's solar telescope is levelled, and the vertical circle is re-set at an

angle of elevation equal to the estimated co-latitude. Shortly before the proposed time of observation, the solar sight or telescope is turned about the polar axis until the sun's image is brought exactly into the centre of the square, or axially on the cross-hairs. The co-latitude is then read on the vertical circle.

(ii) The procedure in this case differs little from that of the preceding paragraph, except that the main telescope is not clamped in the meridian. The solar sight or telescope is brought into the vertical plane of the main telescope ; in Burt's attachment by setting the index at XII on the hour circle, and in Saegmuller's attachment by sighting a common point with both solar and main telescopes. In each case the image of the sun is followed with the solar sights or telescope until it reaches its greatest altitude and begins to descend. The co-latitude is then read on the vertical circle of the theodolite.

Example†. Determine the G.M.T. at which the star α Aurigae crossed the meridian of a station in Longitude 28° 31′ E. in the northern hemisphere at upper culmination on May 31, 1926, the declination of the star being N. 45° 55′ 25″ and its Right Ascension 5 h. 11 m. 6 s., with G.S.T. at G.M.N. 1 h. 32 m. 55 s.

If the true altitude of the star was 76° 30′ 50″, find also the latitude of the station. (U.L.)

L.S.T. of transit				5 h.	11 m.	6·00 s.
G.S.T. (G.M.N.)	4 h.	32 m.	55·00 s.			
Subt. for Long. E.			18·55			
L.S.T. (L.M.N.)	4	32	36·45	4	32	36·45
S.I. after L.M.N.				0	38	29·55
Subtract for M.T.						06·28
L.M.T. of culmination (p.m.)					38	23·27
Long. diff. E.				1	53	00·00
Before G.M.N.				1	14	36·73

or G.M.T. 10 h. 45 m. 23·27 s., May 31st.

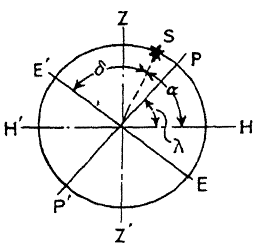


FIG. 128.

$$\lambda = \alpha - (90° - \delta)$$
$$= 76° \ 30' \ 50'' - 44° \ 04' \ 35''$$
$$= \text{N. } 32° \ 26' \ 15''.$$

Example†. Explain the terms " standard time " and " equation of time ".

The following notes refer to an observation on the upper limb of the sun when it was crossing the meridian of a station in the Northern Hemisphere on Sept. 21, 1937 :

(*a*) Observed altitude, 36° 0′ 20″.

(*b*) Time by G.M.T. chronometer, 12 h. 16 m. 10 s.

(*c*) Sun's semi-diameter, 16′ 0″.

(*d*) Declination at G.M.N., N. 0° 46′, decreasing 0·97″ per hour.

(*e*) Equation of time at G.M.N., 6 m. 50 s. subtractive from apparent time and increasing 0·88 s. per hour.

(*f*) Correction for refraction, 57″ cot α, where α is the apparent altitude.

(*g*) Correction for horizontal parallax, 9″.
Find the latitude and longitude of the station. (U.L.)

(1) Mean observed altitude, 36° 00′ 20·00″
Corrections : (− 16′ 00″ − 57″ cot 36° 00′ 20″
+ 9″ cos 36° 00′ 20″) − 17′ 11·15″

True altitude of sun's centre, α = 35° 43′ 08·85″

(2) Declination at G.M.N. N. 0° 46′ 00″
Less 16 m. 10 s. at 58·2″/hr. 15·68″ δ = 45′ 44·32″

$\alpha - \delta$ = 34° 57′ 24·53″

Latitude = 90° − (34° 57′ 24·53″) = 55° 02′ 35·47″ N.

(3) L.A.T. at G.M.T. 12 h. 16 m. 10 s. 12 h. 00 m. 00·00 s.
E.T., 6 m. 50 s. + 16·16 m. at 0·015 s. = 06 50·24

Hence G.M.T. 12 h. 16 m. 10 s. corresponds to L.M.T. 11 53 09·76

and place is 23 m. 0·24 s. W. = 5° 45′ 03·6″ W.

Example†. The following notes were recorded in a zenith-pair meridian observation for latitude :

Star	Zenith distance	Declination	Level	
			Object	Eye
θ	S. 57° 0′ 30″	64° 56′ 41″ S.	6·7	3·3
λ	N. 46° 2′ 00″	12° 2′ 16″ S.	3·5	6·5

Determine the latitude of the place, given that the correction for refraction is 57″ cot altitude and that 1 division of the bubble on the vernier frame is equivalent to 15 seconds.

State clearly what three possible errors should be eliminated or reduced by the double observation. (U.L.)

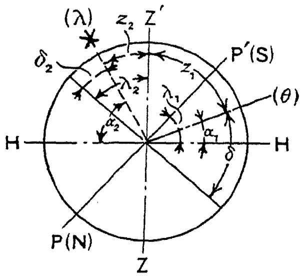


FIG. 129.

Obs. altitude of θ	32° 59′ 30·0″
Bubble : $+15'' \times 1{\cdot}7$	+25·5
	32° 59′ 55·5″
Refraction 57″ cot 33°	−01′ 27·8″
True altitude, α_1	32° 58′ 27·7″
Obs. altitude of λ	43° 58′ 00·0″
Bubble : $-15'' \times 1{\cdot}5$	−22·5
	43° 57′ 37·5″
Refraction 57″ cot 44°	−59·0″
True altitude α_2	43° 56′ 38·5″

†. The latitude of a station 4° 20′ E. of the 120° W. meridian was determined by reducing an observation of β Aquilae to meridian, the true altitude of the star being 39° 20′ 30″ and the approximate latitude of the station 56° 54′ 30″ N.

The time of the observation, 10 h. 55 m. 30 s., was taken with a mean time chronometer, which was 1 m. 25 s. fast on the standard time of the 120° meridian. The R.A. and declination of the star were respectively 19 h. 52 m. 16 s. and 6° 15′ 02″ N., G.S.T. at G.M.N. being 8 h. 30 m. 20 s.

Determine the exact latitude by applying the circum-meridian correction mC to the observed altitude, given that

$$C = \frac{\cos\delta \cdot \cos\lambda}{\cos\alpha} \quad \text{and} \quad m = 2\sin^2\tfrac{1}{2}\omega$$

with the hour angle ω in sec. of arc. (U.L.)

L.S.T. of transit = R.A.					19 h.	52 m.	16·00 s.
G.S.T. (G.M.N.)	8 h.	30 m.	20 s.				
Acceleration		01	16	$\frac{115{\cdot}67^c}{15} \times 9{\cdot}86$ s.			
L.S.T. (L.M.N.	8	31	36		8	31	36·00
S.I. after L.M.N.					11	20	40·00
Retardation 11·389 h. at 9·83 s.						01	51·95
M.T.I. after L.M.N.					11	18	48·05
Corrected S.M.T. of observation	10 h.	54 m.	05 s.				
Correction for 4° 20′ Long. E.		17	20				
L.M.T. of observation	11	11	25		11	11	25·00
M.T. interval before transit						7	23·05
Acceleration ($7 \times 0{\cdot}164 + 23 \times 0{\cdot}003$)							01·21
S.I. before transit = 444·3 s.						7 m.	24·26 s.

Since 1 sec. time $t = 15''$ arc,

$$m = \frac{2 \sin^2 \frac{1}{2}(15t)}{\sin 1''} = \frac{225t^2 (\sin 1'')^2}{2 \sin 1''}$$

$$= \frac{225t^2}{2 \times 206265} = \frac{t^2}{1834} \text{ with } t \text{ in sec.}$$

Thus $m = \dfrac{(444 \cdot 3)^2}{1834} = 107 \cdot 6''$,

$C = \dfrac{\cos \delta \cos \lambda}{\cos \alpha}$,

$mC = 1'\ 15 \cdot 51''$

$\log \cos \delta$	$= \bar{1} \cdot 9974105$
$\log \cos \lambda$	$= \bar{1} \cdot 7371773$
	$\bar{1} \cdot 7345878$
$\log \cos \alpha$	$= \bar{1} \cdot 8883926$
$\log C$	$= \bar{1} \cdot 8461952$
$\log m$	$= 2 \cdot 0318123$
$\log mC$	$= 1 \cdot 8780075$

Correct altitude in meridian

$$= \alpha_0 = \alpha + mC = 39^\circ\ 20'\ 30'' + 1'\ 15 \cdot 51'' = 39^\circ\ 21'\ 45 \cdot 51'',$$

$$\lambda = 90^\circ - \alpha_0 + \delta = \text{N. } 56^\circ\ 53'\ 16 \cdot 49''.$$

Example†. The following extra-meridian observation for latitude was made on Capella from a station A of a survey near Toledo, Spain, in longitude 4° W., the times being kept with a G.M.T. chronometer.

The following data were recorded at the time :

Corrected true altitude of star, 55° 51′ 11·36″.

Corrected G.M.T., 0 h. 50 m. 27·5 s., Jan. 16.

Declination : 45° 55′ 25″ N. ; R.A., 5 h. 11 m. 06 s.

G.S.T. (G.M.M.) 7 h. 35 m. 11.06 s., Jan. 15.

Mean clockwise horizontal angle of star from survey line AB,

162° 22′ 24″.

Determine the azimuth of the survey line AB and the latitude of the station A.

				h.	m.	s.
L.S.T. of transit (R.A. + 24 h.)				29	11	06·00
G.S.T. (G.M.M.)	7 h.	35 m.	11·06 s.			
Acceleration for 4° W.			02·63			
L.S.T. (L.M.M.)	7	35	13·69	7	35	13·69
S.I. between L.M.M. and transit				21	35	52·31
(L.M.T. 0 h. 34 m. 27·5 s.)						
S.I. between L.M.M. and Obs.				24	38	29·75
Hour angle SPZ				3	02	37·44
,, ,, ,,				45°	39′	21·60″

(a) $$\sin SZP = \frac{\sin SPZ}{\sin SZ} \cdot \sin SP.$$

Hour angle	$SPZ = 45° \; 39' \; 21{\cdot}60''$;	log sin	$\bar{1}{\cdot}8544009$
Co-dec.	$SP = 44° \; 04' \; 35{\cdot}00''$;	log sin	$\bar{1}{\cdot}8423700$
Co-alt.	$SZ = 34° \; 08' \; 48{\cdot}64''$;	log cosec	$0{\cdot}2507920$
Sun's azimuth, SZP		log sin	$\bar{1}{\cdot}9475629$

SZP, 62° 24′ 26·70″ counter-clockwise from N.

Azimuth, $AB = 360° - (62° \; 24' \; 26{\cdot}70'' + 162° \; 22' \; 24{\cdot}00'')$
$= 135° \; 13' \; 09{\cdot}30''$.

(b) Now that the sun's azimuth is known, the latitude may be calculated from

$$\tan \tfrac{1}{2}(ZP) = \frac{\sin(SZP + SPZ)}{\sin(SZP - SPZ)} \tan \tfrac{1}{2}(SP - SZ).$$

$' \; 24' \; 26{\cdot}7'' \; + 45° \; 39' \; 21{\cdot}6'') = 52° \; 01' \; 54{\cdot}15''$;	log sin	$\bar{1}{\cdot}8967188$
$\tfrac{1}{2}(62° \; 24' \; 26{\cdot}7'' \; - 45° \; 39' \; 21{\cdot}6'') = 8° \; 22' \; 32{\cdot}55''$;	log cosec	$0{\cdot}8366492$
$\tfrac{1}{2}(44° \; 04' \; 35{\cdot}00'' - 34° \; 08' \; 48{\cdot}64'') = 4° \; 57' \; 53{\cdot}18''$;	log tan	$\bar{2}{\cdot}9388653$
$\tfrac{1}{2}ZP = 25° \; 10' \; 49{\cdot}45''$	log tan $\tfrac{1}{2}(ZP)$	$\bar{1}{\cdot}6722333$

$ZP = 50° \; 21' \; 38{\cdot}90''$. Latitude : 39° 38′ 21·10″.

QUESTIONS ON ARTICLE 4

1†. The following notes were recorded on April 21, 1927, in determining the latitude of a base station on the northern hemisphere, the meridian altitude of Aldebaran being under observation :

Face	Microscopes		Level	
	C	D	O	E
Left - -	51° 40′ 2″	51° 40′ 10″	8	6
Right -	51° 40′ 18″	51° 40′ 12″	4	10

(a) Declination of star on given date, N. 16° 21′ 46″.

(b) Azimuthal level on clipping frame, E and O denoting eyepiece and objective ends respectively. Bubble, 10 sec. per division.

(c) Refraction correction, 57″ cot α, α being the mean apparent altitude.

Reduce the latitude of the station ; and state clearly how the problem would be modified had the star been temporarily obscured, necessitating an extra-meridian observation. (U.L.)

[Lat. 54° 42′ 30·6″ N.]

2†. Determine the G.M.T. at which the star α Andromedae crossed the meridian of a station in long. 36° 12′ E. in the northern hemisphere at upper culmination on June 1, 1940, the declination of the star being N. 28° 45′ 41″

and its right ascension 0 h. 5 m. 17 s., with G.S.T. at G.M.N. 4 h. 35 m. 20 s. on May 31, 1940.

If the meridian altitude of the star was 64° 30′ 50″, find also the latitude of the station. (U.L.)

[5 h. 2 m. 40·42 s. June 1; 54° 14′ 51″]

3††. At a place in longitude 121° 10′ 30″ W. the circum-meridian altitudes of β Aquilae (right ascension 19 h. 52 m. 16 s., declination 6° 15′ 02″ N.) were observed south of zenith for latitude.

Standard time			Face	Vertical angle	
h.	m.	s.		Vernier C	Vernier D
10	53	39 p.m.	R	39° 19′ 30″	0° 19′ 40″
10	56	19	R	39° 21′ 20″	21′ 30″
10	59	20	L	140° 38′ 10″	38′ 30″
11	2	7	L	140° 37′ 50″	38′ 00″
11	5	6	R	39° 23′ 10″	23′ 00″
11	8	3	R	39° 22′ 50″	22′ 40″
11	11	3	L	140° 38′ 30″	38′ 50″
11	13	49	L	140° 39′ 20″	39′ 30″

The standard time is 8 hours slow on Greenwich and the Greenwich sidereal time of Greenwich mean noon was 8 h. 50 m. 21 s.

(a) Calculate the standard time of transit and the approximate latitude.

(b) Make an accurate determination of the latitude employing the circum-meridian correction (to the mean altitude) $\dfrac{\cos\delta\cos\phi}{\cos A}\cdot\dfrac{(\bar{t})^2}{1833{\cdot}5}$ in which δ, and A are the declination, latitude, and meridian altitude respectively and $\bar{t}$ is the interval in sidereal seconds to transit.

It is assumed that the altitude level remained central during the observation, and you may use the approximate value of 58″ cot α for refraction. (U.L.)

[23 h. 03 m. 29·2 s. ; N. 56° 53′ 45·3″.]

4. (a) Two stars, S_1 north of the zenith and S_2 south of the zenith, were observed on the meridian at almost equal altitudes for a determination of a north latitude, by means of a theodolite fitted with a micrometer eyepiece.

The declination of S_1 was 70° 36′ 15″ N. and that of S_2 was 30° 03′ 55″ N. ; also the difference of the altitude readings was 09′ 45″, the altitude of S_1 being the greater.

Find the latitude of the place.

(b) Find the local mean time at which β Leonis (Denebola) made its upper transit on 1 May, 1940, at a place 60° E.

Given : R.A. of β Leonis on 1 May was 11 hrs. 46 mins. 02 sec. and G.M.T. of G.S.N. was 9 hrs. 23 mins. 23 secs. ; also 1 hour S.T. = 1 hour M.T. − 9·8295 secs. M.T. (U.B.)

[(a) 69° 48′ 42·5″, (b) 21 h. 8 m. 08·73 s.]

5. Derive an expression for determining latitude of a place by observing the meridian altitude of the sun or a star.

An observation was made on December 30, 1919, in longitude 82° 17′ 30″ E; the meridian altitude of the sun's lower limb was 40° 15′ 13″. The sun on the south of the observer's zenith. Calculate the approximate latitude of the place.

Correction for refraction 00° 1′ 10″, correction for semi-diam. 00° 16′ 17·5″, correction for parallax 00° 00′ 6·9″. Declination of the sun at G.A.N. was 23° 13′ 15″, decreasing at the rate of 9·17″ per hour. (T.C.C.E.)

[N. 26° 17′ 07·9″.]

6. Derive an expression for determining latitude of a place by observing the meridian altitude of the sun or a star.

An observer in latitude 29° 52′ N. has the line of sight of a theodolite placed in the meridian and directed southwards. He observes that the altitude of a star at its upper culmination is 54° 40′ 30″; if the refraction correction is 35″, find the declination of the star. (T.C.C.E.)

[5° 28′ 05·0″ S.]

7. Explain by the aid of sketches how the quantities in the following groups are related to each other:

(i) The R.A. of a star, the hour angle of the star at any instant and the sidereal time at that instant.

(ii) The declination of a star, its meridian altitude at a place and the latitude of that place.

An observer in latitude 52° 36′ 30″ N. has the line of sight of a theodolite placed in the meridian and directed southwards. He observes that the altitude of a star as it makes its upper transit is 56° 58′ 50″. If the refraction correction is 37 secs., find the declination of the star. (T.C.C.E.)

[19° 34′ 43″ N.]

ARTICLE 5: LONGITUDE

The longitude of a place on the earth's surface may be defined as the angle at the pole between the *standard meridian*, usually that of Greenwich, and the meridian through the place. It is thus equal to the arc of the equator intercepted between these two meridians. Since the earth rotates uniformly about its axis with respect to the mean sun or a star, this angle is proportional to, and is measured by, the time intervening between the respective transits of these bodies. Hence terrestrial longitude may be defined as the difference of the local apparent or mean times of the meridian concerned and the times at the same instant at the standard meridian. Accordingly, longitude is now usually specified in time rather than in arc.

Longitude may be determined (1) by triangulation or (2) by astronomy, the latter including (*a*) chronometer-signal methods, and (*b*) absolute methods.

(1) **Triangulation.** If a suitable triangulation net is projected, this would afford the basis of the most accurate method, though, obviously, its cost would otherwise be prohibitive. Not only would the calculations involve knowledge of the earth's figure, but also they would be exceedingly intricate.

(2) **Astronomy.** (*a*) **Chronometer.** The most general method of determining longitude consists in expressing the difference between Greenwich mean time by chronometer and local mean time as determined by observation of the sun or a star. Usually three or more chronometers were kept, so that the known errors and rates could be balanced. Two or more chronometers are checked to keep the local time of a place X and these are transported to a place Y, and are there compared with chronometers, checked and rated on local time at Y. Considerable difficulty arises in finding the so-called "travelling rate" of transported chronometers, especially in ascertaining if it is uniform.

(*b*) **Signals.** Formerly the telegraph, whenever available, afforded a direct comparison between the observer's chronometer and that of some station of established longitude. Hence the difference of these clocks was the true difference of longitude of the places, after corrections had been applied for the errors and rates of the chronometers, also the personal equation, which was an important factor. The inception of wireless telephony has not only opened up the possibilities of telegraphy to all parts of the earth, but also, through the medium of precise mechanism, affords the basis of the most accurate method, excepting triangulation. For example, a chronometer can be rated to local mean time by paired star observations eight to ten hours apart, and the local time can be compared precisely with the Greenwich time through the medium of the time vernier embodied in the wireless signal (see p. 289).

(*c*) **Absolute methods.** Absolute methods are seldom resorted to in ordinary surveys. In the first place, the instruments usually at the surveyor's disposal would at best lead to approximate values, and then only with laborious calculations, while extremely approximate values of longitude are often sufficient for extra-meridian observations, and even these can be dispensed with by selecting suitable stellar observations for meridian, latitude and time.

For further information on the subject, the student is referred to a standard work on astronomy.

International date line. On account of motion relative to the earth's rotation, a change in date becomes necessary at the 180th meridian from Greenwich. Ships crossing this line *from the east* skip one day in so doing, and Monday afternoon becomes Tuesday afternoon the moment

the date line is passed, the intervening 24 hours being dropped from the reckoning in the ship's log. On the contrary, when a ship crosses the line *from the west,* the same day is counted twice, passing from Tuesday back to Monday, and thence to Tuesday again.

The line adopted by the Admiralty, a modification of the 180th meridian, is traced to include islands of any one group on the same side of the line, and otherwise for political reasons. Eight points in latitudes between 65° N. and 60° S. are selected with longitudes between 169° W. and 170° E., after which the line passes through the centre of Bering Strait to a point in Lat. 70° N. and Long. 180°.

Conversion of arc to time and time to arc. Since $360° = 24$ hours, $15° = 1$ hour, $1° = 4$ minutes ; hence to convert *arc* to *time* :

$$Arc \left\{\begin{array}{lll}\text{Degrees} & \times\, 4 & = \text{Minutes} \\ \text{Minutes} & \times\, 4 & = \text{Seconds} \\ \text{Seconds} & \times\, 4 & = \text{Thirds}\end{array}\right\} Time.$$

To convert *time* to *arc* :

$$Time \left\{\begin{array}{lll}\text{Hours} & \times\, 15 & = \text{Degrees} \\ \text{Minutes} & \times\, 15 & = \text{Minutes} \\ \text{Seconds} & \times\, 15 & = \text{Seconds}\end{array}\right\} Arc.$$

Full conversion tables will be found in Chambers' Tables " and the " Nautical Almanac ".

QUESTIONS ON ARTICLE 5

1. (*a*) Explain how the difference of longitude between two places A and B may be obtained by the exchange of telegraphic signals between two places.

(*b*) Describe the Onogo modified system of wireless time signals and the United States new system of interrupted seconds.

(*c*) A steel tape, supported on the flat, is 100·00 feet long at a temperature of 60° F. and with a pull of 15 pounds. What horizontal distance will it subtend if it is hung over two posts, at the same level, with a pull of 25 pounds and if its temperature is 70° F.?

The weight of the tape is 1·5312 pounds, its coefficient of linear expansion is 0·00000625 for 1° F. and its E is 30×10^6 pounds per square inch.

The weight of a cubic foot of steel is 490 pounds. (U.B.)

[99·9983 ft.]

2. (*a*) Describe the modified rhythmic time signal.

(*b*) Explain the method of obtaining the difference in longitude between two stations by the exchange of telegraph signals, without chronographs.

Why is it necessary to send and receive signals from and by each station in turn and to average results?

(*c*) A steel tape, cross-section 1/300 of a square inch, weight 1·75 pounds, is exactly 150 feet long between its marks at a temperature of 55° F., when stretched with a pull of 10 pounds, with no sag.

If it is stretched between two posts, approximately 150 feet apart, with a pull of 35 pounds and at a temperature of 75° F., and if scratches are made on the lead plates at the tops of the posts, opposite the end marks of the tape, calculate the true distance between the scratches on the posts. The tops of the posts are level.

Given Young's modulus for the tape $= 30 \times 10^6$ pounds per square inch and coefficient of linear expansion $= 0 \cdot 0000065$ for 1° F.

Reduce the distance between the scratches to the value it would have at mean sea level.

Assume that the radius of the earth is 3,956 miles and that the height of the posts above mean sea-level is 2,640 feet. (U.B.)

[150·0414 ft.; 150·0224 ft.]

SECTION V

GEODETICAL SURVEYS

INTRODUCTION

Although the title of the present section might imply that the subject matter is restricted to *geodetical* surveys, as distinct from *plane* surveys, it should be understood that the distinction is fundamentally one of extent of area rather than of operations, since the precise methods of geodesy are followed in the field work of extensive plane trigonometrical surveys. In fact, precise surveys of the latter category are sometimes incorrectly styled geodetic surveys, though, strictly, however extensive the area and however precise the methods, the survey is plane so long as the geodetical considerations of curvature are ignored.

The limit of error determines the extent beyond which a survey must be treated by geodetical methods. American surveyors advance as a rough estimate 100 square miles, even though they make plane computations in respect of areas of much greater extent when no great degree of accuracy is required.

Hence the first part of this section will be devoted to precise methods, which are subject to the modifications imposed by the foregoing considerations. The second part of the section will be devoted to the computation of spherical triangles and problems involving the figure of the earth.

Approximations. Characteristic of the calculations in the present connection is the use of the approximations $\sin\alpha = \tan\alpha = \text{arc}\ \alpha$ for very small angles, these being the first approximation to the expansion of the trigonometrical functions. The upper limits of the angles are $3^\circ\ 8'$ for a ratio of accuracy of 1 : 1,000 ; $1^\circ\ 0'$ for 1 : 10,000 ; $19'$ for 1 : 100,000 ; and $6'$ for 1 : 1,000,000 in the sine approximation, and roughly two-thirds of these angular values in the tangent approximation.

In calculations involving the logarithms of $\sin\alpha$, $\tan\alpha$, and arc α, this form must be adopted in order to avoid the inaccuracies of proportional interpolation between minutes for the intermediate seconds of small angles, since the logarithms change exceedingly rapidly as they approach $-\infty$.

Now to a high degree of accuracy, the arc of a very small angle is $\dfrac{''}{206{,}264{\cdot}8}$, the denominator being the number of seconds in a radian.

Otherwise, $\log \text{arc } \alpha'' = \log \alpha'' - 5{\cdot}3144251$, the numerical value being $\log \text{cosec } 1''$. Hence

$$\log \sin \alpha'' = \log \tan \alpha'' = \log \text{arc } \alpha'' = \log \alpha'' + \bar{6}{\cdot}6855749,$$

the numerical value, the *constant number*, being

$$\log \tan 1'' = \log \sin 1'' = \log \text{arc } 1''.$$

In " Chambers' Tables " the use of logarithmic sines and tangents of small angles is given in the " Explanation ", the process being for a higher degree of approximation. These involve the subtraction of $\frac{1}{3} \log \sec \alpha''$ from $\log \sin 1'' + \log \alpha''$ and the addition of $\frac{2}{3} \log \sec \alpha''$ to $\log \tan 1'' + \log \alpha''$, but in many connections, particularly academical problems, the secant term may be omitted. Sometimes in problems $\log \sin 1''$ is given as $\bar{6}{\cdot}6855925$, which follows from the natural sine of $1''$.

Triangulation systems. The basis of a geodetical survey is the triangulation system, or net, which may be (*a*) *simple*, (*b*) *double*, or (*c*) *treble*, according as the net consists of (*a*) a simple chain, as in the Malta Survey; (*b*) a minor system upon a major system, as in the Orange River Survey; or (*c*) tertiary and secondary triangles upon a primary, or principal system, as in the Surveys of the United Kingdom and India.

Triangulation systems may be (I) theoretical or ideal, and (II) actual, the former affording a basis for the analysis of the actual systems.

(I) **Theoretical systems.** Figs. 130, 131 and 132 show four representative systems; the single and double chains, quadrilaterals or interlacing triangles, and polygons with central stations, the triangles being equilateral and the quadrilaterals squares, since these represent respectively the best-conditioned figures.

Consider a theoretical net of N stations with a common length of side D, and let there be s stations in each separate bay or figure, with x individual conditions in each separate figure, n being the characteristic number for the scheme. There will be $N-2$ theodolite stations, and the number of separate figures in the scheme will be $(N-2)(s-2)$, while the number of conditions will be $n = \dfrac{x(N-2)}{s-2}$.

In general, there will be the following conditions:

(*a*) Triangle condition, in that the angles of each triangle will sum up to $180° + \epsilon$.

(*b*) Apex condition, in that the angles around a central station will sum up to $360° + \epsilon$, where ϵ is the spherical excess (see p. 265).

Consequent to these is the " polygon condition ", in that the angles will sum up to $(2N-4)\ 90°$ + spherical excess of the individual triangles.

(*c*) Log sine condition, in that the deduced length of a side should have the same value, whatever parts of the data are used in its calculation: $\Sigma(\log \sin \text{R.H. angles}) = \Sigma(\log \sin \text{L.H. angles})$.

Obviously (*b*) and (*c*) are not applicable when s is less than four.

Three important factors arise when considering the systems enumerated: (i) the number of equations of condition n to be satisfied, (ii) the area A covered by the system, and (iii) the relative amounts of field operations.

(1) *Simple triangles.*

$$n = x\,\frac{N-2}{s-2} = N-2\,; \quad A = \tfrac{1}{2}D^2\,\frac{N-2}{s-2}\cos 30^\circ = \frac{\sqrt{3}}{4}\,D^2(N-2).$$

(2) *Squares with interlacing triangles.* Here the independent conditions are 2 for each triangle, or 4 for the square in Fig. 131.

$$n = 4\,\frac{(N-2)}{(s-2)} = 2\,(N-2)\,; \quad A = \frac{N-2}{s-2}\,D^2 = \frac{D^2}{2}\,(N-2).$$

(3) *Polygons.* Here the number of independent conditions in each hexagon of Fig. 132 is $6+1$.

$$n = 7\,\frac{N-2}{s-2} = \frac{7}{5}(N-2)\,; \quad A = \frac{N-2}{s-2} \times 6 \times \frac{\sqrt{3}}{4}\,D^2 = \frac{3\sqrt{3}}{10}\,D^2(N-2).$$

Thus the system of interlacing squares gives the most equations of condition and is probably the most accurate and reliable method, though the individual triangles are not so well-conditioned as those in the polygonal scheme. Also the polygonal net covers the greatest area for a given number of stations, though the simple chain is the most economical as regards field operations.

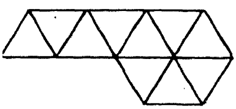


FIG. 130.

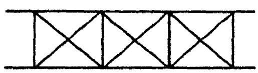


FIG. 131.

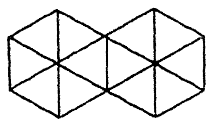


FIG. 132.

(II) **Actual systems.**

These may be said to include the following:

(1) **Rectangular chains.** Chains running approximately parallel and perpendicular to the meridian; or parallel main chains enclosing systems, combinations of which form polygons: France, Spain, Austria, India.

(2) **Oblique chains.** Chains of consecutive triangles; sometimes combinations of triangles forming consecutive polygons: Italy, Sweden, Norway, Russia, Germany, United States of America.

(3) **Irregular nets,** giving polygonal conditions: Malta.

(4) **Interlacing nets,** involving rays additional to those of the individual triangles: United Kingdom, India.

The routine of a geodetical survey normally consists of the following subdivisions, which involve the supplementary operations stated (cf. p. 344).

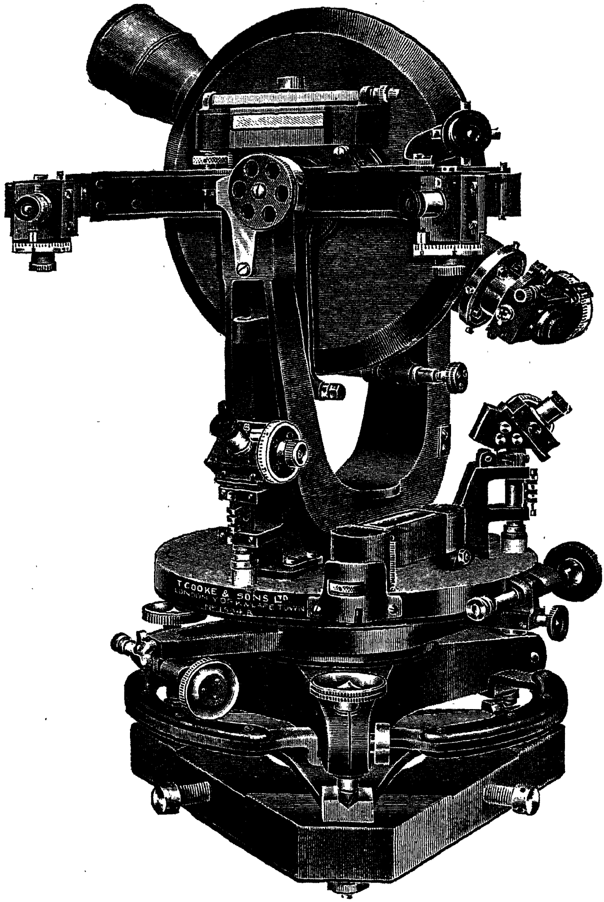


Fig. 133.
Geodetic theodolite.

Preliminary. Adoption, or tentative selection, of triangulation systems, usually based upon departmental reports. Objects, degrees of precision, possible extensions, economic considerations.

(1) *Reconnaissance* with a view to selection of stations, possibly leading to a modification of the proposed scheme. Organisation. Intervisibility, heights of stations, cost, including cutting and clearing, preservation of stations after completion, etc.

(2) *Establishing the triangulation stations.* Scaffolds, signals, and signal apparatus. Ground and satellite stations.

(3) *Measurement of base lines and bases of verification.* Apparatus. Reduction to sea-level, etc.

(4) *Development of triangulation nets.* Instruments and methods of angular measurement (extension of base line), co-ordination of major and minor systems.

(5) *Determination of elevations.* Methods; reciprocal and direct; supplementary spirit levelling.

(6) *Topographical surveys.* Cadastral work within the minor system. Organisation of land surveys personnel.

(7) *Computation and cartographical work.* Errors. Adjustment of triangles, computation of spherical triangles, subdivision of cartographical work, mapping, organisation.

ARTICLE 1: STATIONS, SIGNALS, AND SCAFFOLDS

Stations and scaffolds. Ordinary theodolite tripods are seldom satisfactory for use at the "ground stations" of extensive surveys, though occasionally the heavy patterns may be used. A table improvised with posts and boards, well-braced, is better, the tribrach screws fitting into a special sprang.

When it is necessary to elevate the instrument, a framed triangular scaffold may be used; around this a rectangular scaffold for the observer; and, inside the instrument scaffold, a wooden tube for the plumb line, preferably made to turn so that the leeward side is open.

Numerous types of scaffolds are employed, the form often depending upon local conditions. The scaffolds of the U.S.A. Great Lakes Survey ranged from 10 ft. to 124 ft., with an average height of 58 ft., the signals being 5–30 ft. higher.

Frequently the towers of buildings are utilised, a notable case being that carried up from the tower of Thaxted Church, Essex, in the Ordnance Survey.

Signals. A signal is a device erected to define the exact position of an observed station. Signals, in general, should be conspicuous, giving a well-defined, divisible outline: in construction they should admit of exact centring over the station point.

Signals may be classified as (1) daylight, (2) sun, and (3) night signals.

(1) **Daylight, or non-luminous signals** consist of the various forms of mast, target, and tin cone types, which are normally used for direct sights less than 20 miles distant, though this limit may be increased considerably under favourable conditions.

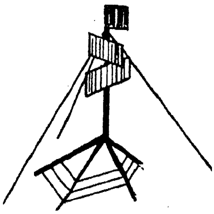


Fig. 134.

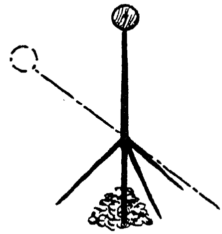


Fig. 135.

Fig. 134 shows a pyramid or quadripod signal with rectangular targets in pairs at right angles, and Fig. 135 shows a circular target of board or framed canvas, surmounting a mast, which may be lowered, as shown, at a ground station. The tin cone is a type which reflects the sun's rays in any direction. Sights up to 45 miles have been taken on these.

(2) **Sun signals** are those in which the sun's rays are reflected to the observing theodolite, either directly, as from a beacon, or indirectly from the signal target. When the distances between stations are very great, special instruments are used : the **heliostat** reflecting a continuous beam of light, and the **heliograph**, periodic beams, as in the case of the Morse code. An early instrument, Gauss' heliotrope, has a counterpart in the Galton sun signal, as made by Messrs. Cooke, Troughton and Simms (Fig. 136), which firm also makes a geodetic pattern of the heliograph.

(3) **Night signals**, as the name suggests, are used in observing the angles of a triangulation system at night. Various artifices have been employed. On the Ordnance Survey, Bengal lights were superseded by the Argand burner with a parabolic reflector, and these latter were

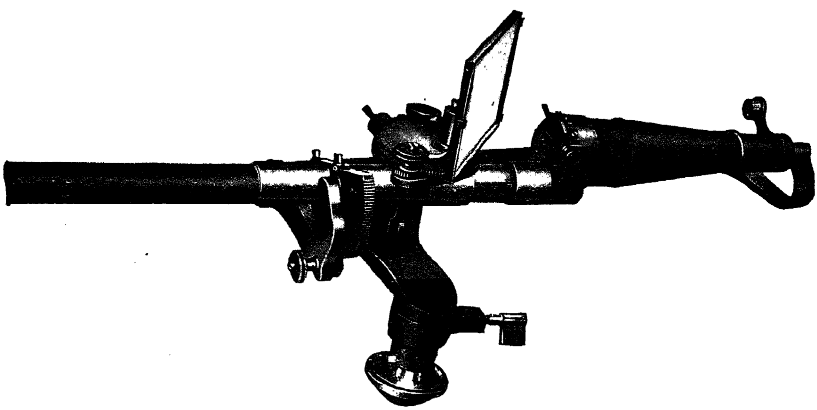


Fig. 136.
Galton sun signal.

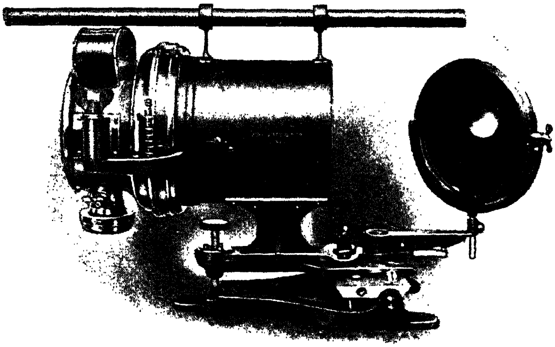


FIG. 137.
Acetylene beacon lamp.

superseded by the plano-convex lens of Fresnel and Arago, as used by Colby and Kater, stations up to 48 miles being observed in this way. The Drummond light had about eighty times the intensity of the Argand burner, being brilliant at about 70 miles in boisterous and hazy weather. Originally this consisted of a ball of lime, $\frac{1}{4}''$ diameter, at the focus of a parabolic reflector, the ball being made incandescent by a stream of oxygen directed through an alcohol flame. Acetylene survey lamps have been used with considerable success in recent years. An excellent complete model is that made by Messrs. E. R. Watts & Son to the designs of Capt. G. C. McCaw, for use on a Boundary Survey Commission. This instrument comprises a lamp of 600 c.p., a mirror and a condenser, the outfit including generator, spares, and all accessories. Attached to the lamp is a 12-power sighting telescope, and on the base plate an optically-worked heliograph mirror. The base affords both lateral and vertical adjustment, while a movement of about 20° in azimuth is obtainable. A circular bubble is provided for levelling (Fig. 137).

Dimensions of signals. Various rules exist as to the diameter of a signal, 3″–8″ being given, according to local conditions. Sometimes 1″ per 10–15 miles is suggested. A height in the vertical plane corresponding to at least 30″ is necessary, while the diameter of the base should be one-third of the height. The following rules may serve as a guide in this connection :

Diameter in ft. $= 0{\cdot}07\text{–}0{\cdot}1D$, with D in miles.
Height ,, $0{\cdot}7D$,, ,,

Phase of sun signals is the error of bisection of certain types of signals which arises from the fact that, under lateral illumination, the signal is partly in light and partly in shade, except when seen against the sky or when the sun is in the plane of the line of sight.

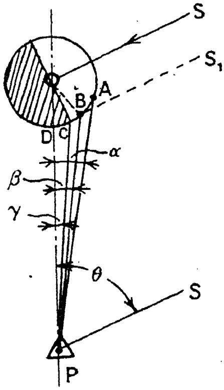


FIG. 138.

The effect is common with cylindrical signals, also with square masts, unless one side faces the theodolite, while with target signals it arises from the shadow of the upper target falling upon the lower. Phase may be avoided by using single targets, which are set normally to the line of sight when the station is under observation; and it may be reduced considerably when the depth of the targets is large in comparison with their widths.

A correction for phase may be made for the cylindrical type in the following manner, according as (*a*) the bright portion is bisected, or (*b*) the bright line is observed, the latter giving more definite corrections.

Consider Fig. 138, where the point B is assumed to move to suit both cases. Here P is the observer's station, O the centre of the signal, and θ the angle the sun makes with PO.

(*a*) The illuminated surface will appear to extend from A to C, and if this be bisected by a sight along PB, the phase correction will be

$$\beta = \tfrac{1}{2}(\alpha - \gamma) + \gamma = \tfrac{1}{2}(\alpha + \gamma).$$

Then if r is the radius of the signal and D is the sight length, α and γ being very small,

$$\alpha = r/D \text{ and } \gamma = r(\cos\theta)/D;$$

whence
$$\beta = \frac{r(1+\cos\theta)}{2D} = \frac{r\cos^2\frac{1}{2}\theta}{D} \text{ rads.},$$

or
$$\beta'' = \frac{r\cos^2\frac{1}{2}\theta}{D\sin 1''} \text{ sec.}$$

(*b*) If, however, observation is made on the bright line formed by the reflected rays, as indicated by the dotted path from S_1 to B, angle S_1BP is $180° - (\theta - \beta)$ and angle PBO is $90° + \frac{1}{2}(\theta - \beta) = 90° + \frac{1}{2}\theta$ sensibly; then by the sine rule and allowable approximations,

$$\beta'' = \frac{r\cos\frac{1}{2}\theta}{D\sin 1'} \text{ sec.}$$

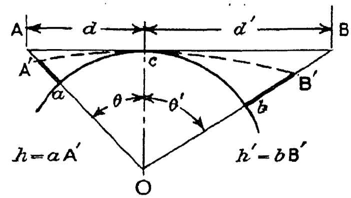


FIG. 139.

Heights of stations or scaffolds. Let a and b be two stations where it is proposed to erect scaffolds of heights h and h' respectively. Let AB be drawn tangential to the earth's mean surface at c in Fig. 139, giving hori-

zontal distances d and d' which correspond with respective angles θ and θ' at the centre of the earth. Now $(Aa+R)^2 = d^2 + R^2$, and since $(Aa)^2$ is negligible in comparison with the other dimensions, the curvature correction is

$$Aa = d^2/2R \text{ very nearly (see p. 192).} \qquad \text{.................(1)}$$

Also, if m be the coefficient of refraction, say 0·07, the corresponding effects will be represented by $AcA' = m\theta$ and $BcB' = m\theta'$; but $\theta = 2Aca$ and $\theta' = 2Bcb$, and since the angles are very small, $AA' = 2m(Aa)$ and

Whence $A'a = Aa - AA' = \frac{a}{2R}(1-2m)$, and if $h = A'a$ be expressed in feet with d and R in miles, $h = 0{\cdot}568d^2$, or

$$d = \sqrt{\frac{h}{0{\cdot}568}}. \qquad \text{..............................(2)}$$

Likewise $d' = \frac{\quad}{0{\cdot}568}$, the corresponding rough rule being $d = \sqrt{1\tfrac{3}{4}h}$ miles/feet.

The actual heights of the signals will be 5 ft.–30 ft. higher.

It should be noted that the foregoing considerations are theoretical in that tangency to mean sea-level is assumed, whereas in practice the calculations are modified by the configuration of the intervening ground. Although the calculations may be made at mean sea-level, they must be transferred to the ground elevations of the stations.

Intervening heights. When peaks and other obstructions occur in the intervening ground, it is necessary to investigate the effect of these upon the heights of the signals, an approximate estimate sufficing on account of the uncertainty of refraction (see Example, p. 329).

Satellite stations. It occasionally happens that angles are observed to a station which cannot afterwards be occupied, as in the case of spires, towers, etc., and, in order to obtain the summation check of the angles having this station as vertex, it is necessary to establish an *eccentric* station conveniently near the unoccupied principal station. The artifice is otherwise known as "reduction to centre", or using a "false station".

Normally the procedure consists in carefully measuring the distance x between the satellite station A_0 and the principal station A; and then observing the angles at A_0, namely, BA_0C and AA_0C, the angle BAC being taken as $180° - (B+C)$ in order that the sides b and c of the triangle ABC can be calculated from the side a, the length of which has been determined from the adjacent triangulation (Fig. 140). From these data the value of the angle A is determinate, the difference $A = 180° - (B+C)$ providing a check.

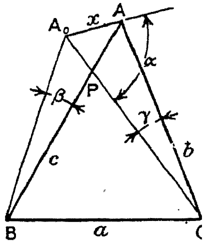


FIG. 140.

Thus, if α is the observed angle AA_0C, $\beta = A_0BA$;

$$A = BPC - \gamma \quad \text{and}$$

Whence $\qquad A = A_0 + \beta - \gamma.$

Now since β and γ are very small, the approximation in circular measure is admissible.

Thus $\qquad \gamma = \sin\gamma = x/b \sin\alpha$;

and $$\gamma = \frac{x\sin\alpha}{b\sin 1''} \text{ sec.}$$

Likewise $\qquad \beta = \sin\beta = \dfrac{x\sin(\alpha + A_0)}{c}$ and $\beta = \dfrac{x\sin(\alpha + A_0)}{c\sin 1''}$ sec.

Therefore $$A = A_0 - \frac{x}{\sin 1''}\left(\frac{\sin\alpha}{b} - \frac{\sin(\alpha + A_0)}{c}\right)$$

When a round of angles is taken from a station A_0 eccentric to, and a distance x from, the central station A, the direction AA_0 is assumed 0° temporarily, and the observed angles are reduced to this zero, being styled conveniently "assumed azimuths" θ. Then the corrections in seconds will follow mechanically with the correct signs from the expression $\dfrac{x\sin\theta}{d\sin 1''}$, where d is the distance of the observed station, being the length of the side opposite θ.

Frequently it is necessary to determine the eccentric distance x (i) by observing the vertical angles to A from A_0 and another point at a known distance from A_0 and also in the same vertical plane as A, or (ii) by determining x by triangulation with A_0 as a station, though (iii) occasionally the distance may be determined by taking tangential sights to the tower, etc., and bisecting the included angle.

The following are the approximate distances and elevations in feet of five triangulation stations on eminences around a central station O, which is a ground station, 740 ft. above datum:

Station	-	P	Q	R	S	T	
Distance	-	18,420	19,650	15,840	24,620	21,650	ft.
Elevation	-	964	752	720	844	1,027	ft.

Given that the instrument is 4·5 ft. above ground stations and the 700 contour is the lowest in the area, determine:

(a) the heights of the scaffolds wherever necessary;

(b) the heights and diameters of the signals for daylight observations.

Also, in the case of R, calculate the correction for phase when the bright line is observed, the sun making an angle of 60° with the line OR.

The earth's mean radius may be taken as 3960 miles, the coefficient of refraction 0·07, and sin 1″, $4{\cdot}85 \times 10^{-6}$. (U.L.)

Here $$h = \frac{a^2}{2R}(1 - 2m) = \frac{a^2}{2 \times 3960} \times 0{\cdot}86 \times 5280.$$

$$\therefore \ d = \sqrt{\frac{h}{0{\cdot}573}} \text{ mls./ft.}$$

Thus at sea-level a sight of $d = \dfrac{4{\cdot}5}{0{\cdot}573} = 2{\cdot}8$ mls. $= 13{,}620$ ft. is possible from a ground station, and thus a total distance of 27,240 ft. is possible, provided the ground is not lower than 740 ft. Hence all stations are ground stations with the exception of R. For R, there is an excess distance of $(15{,}840 - 13{,}620)$ ft. $= 0{\cdot}42$ mls., and the corresponding height of R at sea-level would be 0·10 ft. By a concentric arc at 740, the height at R should be $(740 - 720) + 0{\cdot}1 - 4{\cdot}5$, say 16 ft.

Using the respective rules $0{\cdot}1d$ and $0{\cdot}7d$ for diameter and height of signal in ft., d being in miles :

d -	3·5	3·72	3·00	4·66	4·10 mls.
Diameter	0·35	0·37	0·30	0·47	0·41 ft.
Height -	2·45	2·46	2·10	3·26	2·87 ft.

When the bright line is sighted, the phase correction

$$\beta'' = \frac{r \cos \frac{1}{2}\theta}{d \sin 1''} = \frac{1{\cdot}25 \cos 30^\circ}{12 \times 15840 \times 4{\cdot}85 \times 10^{-6}} = 1{\cdot}18''.$$

Example†. The altitudes of two proposed triangulation stations A and C, 65 miles apart, are respectively 703 ft. and 3520 ft. above sea-level datum, while the heights of two eminences B and D on the profile between A and C are respectively 1170 and 2140 ft., the distances AB and AD being respectively 24 miles and 45 miles.

Ascertain if A and C are intervisible, and, if necessary, determine a suitable height for a scaffold at C, given that A is a ground station. The earth's mean radius may be taken as 3960 miles, and the coefficient of refraction 0·07. (U.L.)

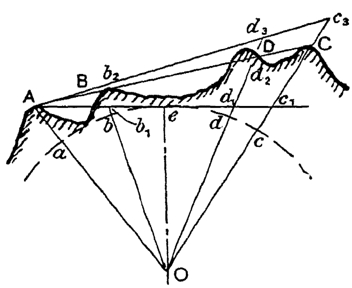


FIG. 141.

Assuming a horizontal sight through A in accordance with

$$\sqrt{\frac{h}{0{\cdot}574}} = 35 \text{ mls.}$$

Thus, $be = 11$ mls. ; $ed = 10$ mls. ; $ec = 30$ mls. (Fig. 141).

Whence by the formula :

$$bb_1 = 59\cdot4 \text{ ft.} ; \quad dd_1 = 57\cdot4 \text{ ft.} ; \quad cc_1 = 516\cdot4 \text{ ft.}$$

Testing if a line of sight between A and C will clear (a) Peak B, (b) Peak D :

(a) $$b_1b_2 \_ Ab_1 \_ 24$$

But $$c_1C = 3520 - 516\cdot4 = 3003\cdot6 \text{ ft.};$$
and $$b_1b_2 = \tfrac{24}{65} \times 3003\cdot6 = 1109\cdot0 \text{ ft.}$$

Thus the elevation of the line of sight at B is $69\cdot4 + 1109\cdot0 = 1178\cdot4$ ft., and Peak B is cleared by $8\cdot4$ ft.

(b) Likewise $\frac{{}_1d_2}{c_1C} = \frac{Ad_1}{Ac_1} = \frac{45}{65}$; $\quad {}_2 = \frac{45}{65} \times 3003\cdot6 = 2079\cdot4$ ft.

Elevation of line of sight at D is $57\cdot4 + 2079\cdot4 = 2136\cdot8$ ft., and this fails to clear by $2140 - 2136\cdot8 = 3\cdot2$ ft.

Assuming a clearance of 10 ft. at D, $d_2d_3 = 10 + 3\cdot2 = 13\cdot2$ ft. ;

$$\frac{}{Cc_3} = \frac{45}{65}, \text{ or } Cc_3 = \frac{65}{45} \times 13\cdot2 = 19\cdot07 \text{ ft.},$$

which is the minimum height of a station at C.

Example†. On occupying a ground station O of a triangulation survey, it was evident that some elevation of the theodolite would be necessary, in order to sight the signals at adjacent stations ; P on the left and Q on the right. It was found, however, that these stations could be seen from a ground station o, south-west of O, so that the line oO approximately bisected the angle PoQ.

Whereupon o was adopted as a false station and the distance oO was carefully measured, being 9 ft. 3·6 in., while the angles PoO and OoQ were observed to be 28° 16′ 35″ and 31° 22′ 20″ respectively. The side PQ was computed to be 3264 ft. in the adjacent triangle, and when the signal at O was under observation, the interior angles at P and Q were found to have mean values of 62° 34′ 15″ and 57° 39′ 20″ respectively. Determine accurately the magnitude of the angle POQ. (U.L.)

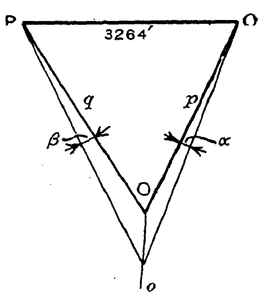


FIG. 142.

Let $oPO = \beta$; $oQO = \alpha$; $oO = 9\cdot3$ ft. (Fig. 142).

Assuming no angular error, and writing

$$POQ = 180° - 120° \, 13' \, 35'' = 59° \, 46' \, 25''.$$

Then by the sine rule :

$$p = \frac{3264 \sin 62° \, 34' \, 15''}{\sin 59° \, 46' \, 25''}$$

log 3264	$=3{\cdot}5137502$
log sin 62°, etc.	$\bar{1}{\cdot}9482080$
	$3{\cdot}4619582$
log sin 59°, etc.	$\bar{1}{\cdot}9365354$
log p	$3{\cdot}5254228$

Similarly : $q = \dfrac{3264 \sin 57° \, 39' \, 20''}{\sin 59° \, 46' \, 25''}$.

log 3264	$=3{\cdot}5137502$	log 9·3	$=0{\cdot}9684829$
log sin 57°, etc.	$\bar{1}{\cdot}9267781$	log sin 31°, etc.	$=\bar{1}{\cdot}7165006$
	$3{\cdot}4405283$		$0{\cdot}6849835$
log sin 59°, etc.	$\bar{1}{\cdot}9365354$	log p	$3{\cdot}5254228$
log q	$3{\cdot}5039929$	log sin α	$=\bar{3}{\cdot}1595607$
		log sin 1″	$\bar{6}{\cdot}6855749$
		log α''	$2{\cdot}4739858$

$\alpha = 4' \, 57{\cdot}84''$.

In triangle oQO,

$$\sin \alpha = \frac{9{\cdot}3 \sin 31° \, 22' \, 20''}{p}$$

In triangle oPQ,

$$\sin \beta = \frac{9{\cdot}3 \sin 28° \, 16' \, 35''}{q}.$$

log 9·3	$0{\cdot}9684829$
log sin 28°, etc.	$\bar{1}{\cdot}6755266$
	$0{\cdot}6440095$
log q	$3{\cdot}5039929$
log sin β	$\bar{3}{\cdot}1400166$
log sin 1″	$\bar{6}{\cdot}6855749$
	$2{\cdot}4544417$

$\beta = 4' \, 44{\cdot}74''$.

Corrected angle

$$POQ = PoQ + \alpha + \beta = (28° \, 16' \, 35'' + 31° \, 22' \, 20'') + 9' \, 42{\cdot}58''$$
$$= 59° \, 48' \, 37{\cdot}58''.$$

(Error from use of tabular values would be 1·44″.)

QUESTIONS ON ARTICLE 1

1†. Two points A and B, 42 miles apart, are selected for triangulation stations, the estimated elevations above datum being 74·0 and 739·0 respectively. There is no intervening high ground between A and B, the lowest points being 50 ft. above datum. A scaffold 25 ft. in height has been selected for A. Find a suitable height for the scaffold at B, also appropriate height and diameter for the signal at this station.

[Scaffold, 35 ft. ; signal, 2·38 ft. diam. ; 24 ft. in height]

2†. Discuss the effect of "phase" in sighting a sun signal, and show with sketches how it may be eliminated or reduced.

Derive formulae for the correction to be applied to cylindrical signals, (*a*) when the *bright portion* is seen bisected, and (*b*) when the *bright line* is observed. (U.L.)

3†. In observing the angles of a triangle ABC, it is found necessary to establish a satellite station A_0 near A, the signal at A being observed from B and C, giving the interior angles at B and C respectively 63° 41′ 25″ and 57° 24′ 15″. The angle at A_0 subtended by B and C is then observed, being 53° 26′ 30″, while the angle AA_0C is 74° 15′ 35″. The side BC was computed to be 1865 ft. in the adjacent triangle, and the distance AA_0 is 5 ft. 4·8 in. Determine the magnitude of the angle BAC. [53° 25′ 1·16″.] (U.L.)

4††. The following notes refer to observations involving a ground station O near a main triangulation station C on a water tower, the heights of the instrument in Form I being read with a horizontal sight on a staff near the railing round the tower.

FORM I

Station	Sighting	Mean vert. angle	Height instr.	Hor. dist. (ft.)
O	C	22° 52′ 00″	4·60 ft.	OP 85·0
P	C	31° 30′ 00″	4·40 ft.	
OPC in same vertical plane				

FORM II

Station	Sighting	Mean hor. angle	Computed dist. (ft.)
O	A	0° 0′ 0″	$AB = 25{,}820$ $AC = 22{,}440$
O	B	61° 48′ 20″	
O	C	308° 24′ 40″	

Determine the angle ACB. (U.L.)

[Eccentric dist. OC, 273·66 ft. ACB, 61° 47′ 10·7″ with log sin 1″ $= \bar{6}{\cdot}6855895$]

5†. Directions are observed from a satellite station S, 204 ft. from station C, with the following results:

A, 0° 0′ 0″; B, 71° 54′ 32″; C, 296° 12′ 2″.

The approximate lengths of AC and BC are respectively 27,036 ft. and 35,642 ft. Calculate the angle ACB. (U.L.)

[71° 44′ 59″]

6†. Explain the reasons for using a satellite station during a trigonometrical survey.

From a satellite station, D, angles were measured to three trigonometrical stations A, B and C and recorded as follows :

Angle ADC	69° 14′ 27″.
Angle ADB	76° 29′ 47″.
Distance DB	64·63 ft.

The stations C and D were on opposite sides of the line AB. The approximate lengths of AB and BC were 16,246 ft. and 19,321 ft. respectively. Determine the angle ABC. (U.L.)

[69° 26′ 18″]

7. X is a station of a triangulation survey to which observations have been taken from adjacent stations. It is impracticable to observe from X, and the theodolite is placed at Y near X. Show how angles observed from Y may be adjusted to yield the values which would have been obtained at X. (I.C.E.)

8. Write notes on *three* of the following :

(*a*) The two-point problem in plane table work.
(*b*) Signals used in triangulation.
(*c*) Satellite station.
(*d*) Corrections to be applied in base line measurements. (U.G.)

9. Derive the approximate formula generally used for reducing observations at eccentric stations to the centre.

Find the effect of the eccentricity of a theodolite at a station in a traverse which is not straight.

For an eccentric station 20 ft. from the centre find at what distance the use of the approximate formula will result in an error of one second of arc in the reduced angle.

Horizontal circle observations at an eccentric station 10 ft. from A to B, C and A were : 0° 0′ 0″ ; 204° 15′ 43″ and 35° 20′ respectively. Find the reduced angle BAC correct to the nearest second of arc. (U.C.T.)

10. (*a*) A beacon over which it is impossible to set up has to be fixed in a farm survey.

Describe three distinct methods of fixing this beacon, and state with reasons what constitute favourable conditions for each method.

(*b*) The top of a flag has been sighted from several other points. When arriving to take observations at this flag, it is found to be leaning. What should be done in order to get the best results from the observations taken? (U.C.T.)

11. Explain carefully what is meant by an eccentric station and eccentric correction (or reduction to centre).

Hence reduce the following observations to centre and state the final values of the directions if AB is known to be 203° 24′ 14″.

	At A eccentric observed direction	Distance
B - -	16° 24′ 08″	5200 ft.
C - -	124° 17′ 43″	3000 ft.
D - -	250° 39′ 16″	824 ft.
E - -	310° 41′ 29″	7300 ft.
A - -	2° 00′ 40″	6·25 ft.

Prove the formula you have used. (U.C.T.)

[*B*, 217° 48′ 43·6″; *C*, 325° 47′ 20·3″; *D*, 91° 38′ 32·9″; *E*, 152° 02′ 45·1″; *A*, 203° 24′ 14·0″. See p. 328.]

ARTICLE 2: BASE LINE MEASUREMENT

Base line. A base line is a line of considerable length, laid down with great accuracy of measurement and alignment as the known side from which, in conjunction with a series of angular observations, is computed the system of triangulation forming the basis of:

(i) an extensive plane trigonometrical survey, or

(ii) a geodesical operation, such as the measurement of an arc of the meridian, or of a parallel, or for the formation of a geographical map.

Methods. Base lines are measured in either the (1) **long-length** or the (2) **short-length** systems.

The former include the apparatus of Jaderin, Wheeler, Guillaume-Carpentier, etc.; the latter, that of Colby, Bessel, Borda, Struve, Ibanez, etc. The history of base line measurement provides one of the most fascinating chapters in the annals of geodesy: one unfortunately outside the scope of a work of this nature.

Since the meridional observations of Snellius in 1615, there have been alternating, though scarcely comparable vogues of both the long- and short-length systems. Roy's pine and glass rods (1784) were superseded by Ramsden's 100 ft. chain (1791) on the Ordnance Survey of the United Kingdom. The vagaries of temperature effects then attracted the interest of scientific men; and a new era in the short-length system was created by the introduction of Borda's differential bars, which were used in the determination of a meridional quadrant as the basis of the metric system in 1805. Colby's iron and brass bars were adopted by the Ordnance

Survey in 1826, and Bessel's iron and zinc rods were used to connect the Central European triangulation chains with that of Russia, exceedingly fine work being carried out on the Baltic shores in 1836.

Now the long-length system is favoured, though the use of bars is not altogether superseded. Jaderin, in 1889, demonstrated the possibilities of long lengths with his 25-metre steel and brass wires, measuring lines with a probable error of the mean of 1 in 10^6. Wheeler (1890) obtained an even higher degree of accuracy with a 300 ft. steel tape. The introduction of nickel steel alloys in 1906 led to the manufacture of low-coefficient tapes, opening up the possibilities of the more adaptable and economical long-length methods.

Steel tape measurements. The observations of Wheeler on the Missouri River Survey demonstrate the accuracy attainable with an extremely simple apparatus. A brief description of his method will suggest to the student means of laying down base lines for precise engineering work and trigonometrical surveys of moderate extent.

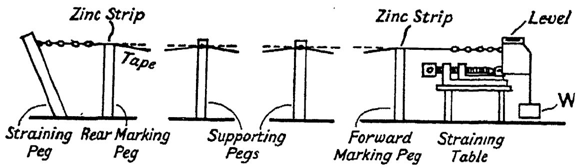


FIG. 143.
Wheeler's base line apparatus.

At distances apart slightly less than the length of the tape, " marking " pegs are driven on the line with their tops about 2 ft. above the surface of the ground. Strips of zinc, $1\frac{1}{2}$ in. in width, are nailed to the tops of the pegs for the purpose of scribing off the extremities of the tape. At intervals of 20–50 ft., light stakes are driven with their faces in line. Nails in these carry hooks to support the tape. These hooks are so placed that the points of support are on either a uniform gradient or the same level. In front of the forward marking peg, three " table " pegs are fixed to support the straining apparatus. These are driven at such a height that the tape rests lightly on the strips during straining.

The **straining device** consists of a frame in which a slide is moved by means of a screw. The upper forward edge of the slide affords a knife-edge fulcrum to the rear lower edge of a block. A spirit level is fixed on the top of this block and a weight is suspended from the lower forward edge. The end of the tape is connected with the upper rear edge of the block by means of a piece of light chain. The dimensions of the block are such that, when the tape is laid for measurement, the tension P on the tape is equal to the applied weight W, the upper surface of the block

being level, as indicated by the spirit level. That is, by the principle of moments :

$$Ph = Wh = Wb + wd, \quad \text{and} \quad h = b + (w/W)d,$$

where h is the vertical height of the hook above the fulcrum ; b, the horizontal distance from the point of application of the weight ; w, the weight of the block, and d, the horizontal distance from the fulcrum to the vertical through the centre of gravity of the block.

Behind the rear marking peg a "straining" peg is driven. The rear end of the tape is attached to a slide on the latter peg ; and the rear graduation of the tape is brought into coincidence with a mark on the zinc strip of the rear marking peg by means of the adjusting screw of the slide. When this adjustment has been effected, and the block on the table has been levelled by means of the screw of the slide, the forward end of the tape is marked on the zinc strip of the forward marking peg. The thermometers are then observed, and the tape is advanced for another length, and so on. A check measurement is made by going over the line again in the *same* direction, the marking pegs and strips remaining for this re-measurement.

Improvised apparatus. Occasions often arise in which the initial outlay on permanent apparatus is not warranted, and in this connection it can be stated that measurements up to 1 : 750,000 may be made with some modification of Wilson's apparatus. Or failing an invar tape, the difficulties concomitant with temperature changes may be avoided by the conjoint use of (say) 21 B.S.G. steel and 20 B.S.G. brass wires, using standard and field tensions so that the sagging effects are the same. Smaller section square pegs may be used if the transverse mark is made on the strip on scribing the longitudinal line when fixing the alignment, plus or minus corrections to the tape length being measured by means of a steel rule or vernier calliper. Better still, a number of strips of zinc, say 3 in. $\times$ 1 in., may be divided into hundredths or two-hundredths of a foot, and a short length of the tape or attached flexible sleeve is divided into appropriately smaller divisions to serve as a vernier.

A weighted device is usually better than the spring balance, which, however, is easier to manipulate, and, incidentally, does not introduce errors as great as are often ascribed to it.

Excellent work can be carried out with Chesterman's steel bands, $\frac{1}{8}$ in.–$\frac{1}{4}$ in. in width, the plain lengths being used for the measurements proper and the studded length for inserting the marking pegs and intermediate stakes for supporting the tape against full sag effect. Since only the plain steel portion can be used, the measuring lengths are 4–6 in. short of 100 ft., appropriate marks being cut with a file.

Improvised apparatus usually consists of pegs not exceeding 2 ft. in height. Hard wood pegs could be superseded by ones of H section in duralumin, suitably capped. Tripods must be used on hard ground. In

general, low pegs are to be preferred to tripods when speed is an important factor. A standard steel tape (with a National Physical Laboratory Certificate) should be available ; and wherever possible the standard pull should again be used in the field. The use of " normal " tensions and the like should not be favoured.

Messrs. Cooke, Troughton & Simms make up an apparatus consisting of tripods carrying the straining device, the measuring head, also with telescope and a target, all fitted with every refinement.

Messrs. E. R. Watts & Son manufacture an apparatus that has been used on many notable surveys in the Near East, Congo, China, and Australia, including work in connection with Sydney Harbour Bridge (Figs. 144, 145).

Corrections to base tape measurements. The immediate corrections to be applied to steel tape measurements consist of those due to (1) **temperature**, (2) **elasticity**, (3) **sag**, and (4) **slope**, reduction to sea-level datum being a later consideration. Frequently the first three are called **coefficients**, in uniformity with the coefficient of expansion.

A standard tape or band is one the length of which is certified by an official standard (such as that of a Government Survey or Physical Bureau) under standard conditions of tension and temperature (usually 62° F.) and the *absolute* length of which under other conditions can be reduced to standard by means of standard corrections. Invar standards have largely superseded ordinary steel standards in recent years.

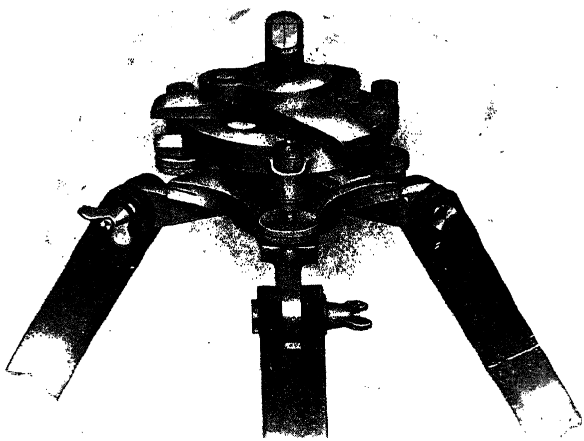


FIG. 144.
Index tripod.

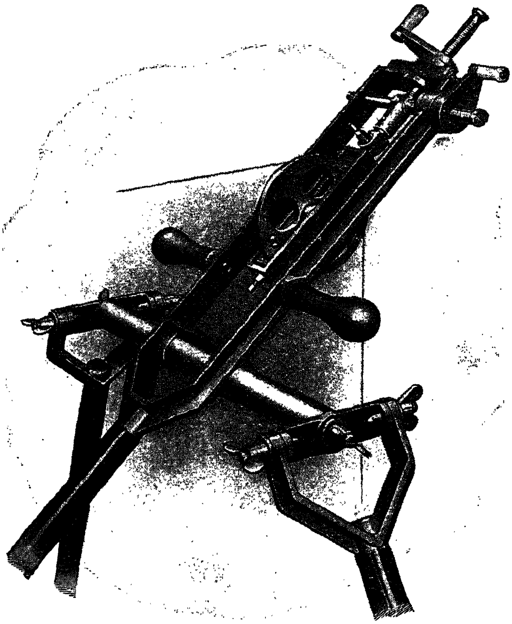


FIG. 145. Straining trestle.

(1) **Temperature effect.** Let P be the standard pull on a tape of standard length L on a plane surface at standard temperature T.

Then the coefficient of expansion $\alpha = \dfrac{l_2 - l_1}{(t_2 - t_1)\,l_1}$ per degree per unit length, where l_2 and l_1 are the lengths of the tape under a constant pull p on a plane surface at respective temperatures t_2 and t_1.

When t_1 is very nearly T and p is very nearly P, the unknown length l_1 may be taken as L, whence :

$$\alpha = \frac{l_2 - l_1}{(t_2 - t_1)\,L} \qquad \text{.........................(1)}$$

is determinable, the difference $l_2 - l_1$ having been observed for the temperature change $t_2 - t_1$.

(2) **Elastic effect.** The increase in length e, due to a pull p, is expressed by $e = pl/AE$, where l is the length, A the cross-sectional area, and E Young's Modulus, which may be taken as 28 (or 30) $\times 10^6$ lb./sq. in. for steel.

Or, uniformly with (1), a *coefficient of stretch* may be used ; namely, $\epsilon = \dfrac{l_2 - l_1}{(p_2 - p_1)\,l_1}$ per unit pull per unit length, where $l_2 - l_1$ is the observed

elongation at a constant temperature for an increase of pull $p_2 - p_1$ on a plane surface. When p_1 is equal to P, the unknown length l_1 may be written as L ; whence

$$\epsilon = \frac{l_2 - l_1}{(p_2 - p_1)L} \quad \text{..............................(2)}$$

is determinable, the difference $l_2 - l_1$ having been observed for the change in pull $p_2 - p_1$ at a constant temperature which differs inappreciably from T.

Observations may be made at any other constant temperature now that the coefficient of expansion is known.

Further, $$E = \frac{\text{stress}}{\text{strain}} = \frac{p_2 - p_1}{bt} \times \frac{l_2 - l_1}{l_1} = \frac{(p_2 - p_1)L}{(l_2 - l_1)\,bt};$$

or $E = 1/\epsilon bt$ nearly, and $\epsilon = 1/Ebt$, where b and t are the respective breadth and thickness of the tape. In general it is advisable to obviate corrections for stretch by standardising the measuring tape under the pull that will be employed in the field.

(3) **Sag effect.** The linear effect of sag x, that is, the difference between d, the horizontal distance between the supports, and l, the length of the catenary affected by an (assumed) inelastic tape is expressed approximately by the parabolic formula :

$$x = (l - d) = \frac{d}{24}\left(\frac{wd}{p}\right)^2 \quad \text{........................} \quad (3)$$

where p is the applied pull and w the weight per unit length of the tape. As in the case of the foregoing, corrections are expressed in pounds and feet. The effect of sag can be determined by (a) weighing the tape and substituting in Eq. (3) ; (b) solving this equation for d ; or (c) calculating d from simultaneous observations with pulls p_1 and p_2 corresponding to observed tape lengths l_1 and l_2, corrected for stretch and temperature.

When d, the distance between the marking pegs, is divided into n equal sags by $(n-1)$ intermediate supports, the net sag effect is

$$\frac{d}{24}\left(\frac{wd}{np}\right)^2.$$

In practice, sagging and stretching are combined in an *elastic* tape, and (being of opposite sign) the latter offsets to various degrees the former effect. The *normal tension*, that is, the pull which exactly counterbalances the sag, is expressed by the relation

$$p_n = \sqrt{\frac{w^2\, d^2\, Ebt}{24 n^2}},$$

where n is the number of sags in d, the distance between the marking pegs.

(4) **Slope effect.** When the supports are not at the same level, the corrected sloping distance is reduced to the horizontal by subtracting

$c = \frac{h^2}{2l}$, where h is the difference in height of successive marking pegs and l is the sloping distance between them, being effectively the tape length in a flat catenary. This relation follows from the first approximation to the expansion of $\sin^2 \frac{1}{2}\alpha$, where α is the angle of slope; and is correct to $1 : 10^3$; $1 : 10^4$; $1 : 10^5$; and $1 : 10^6$ according as the ratio h/l is not greater than 0·25, 0·14, 0·08, and 0·045. That is, the difference in level of the marking pegs can be as great as 4·5 ft. in 100 ft. for a limit of 1 in 10^6. Greater precision would involve the second approximation,

$$c = -\left(\frac{h^2}{2l} + \frac{h^4}{8l^3}\right).$$

Although *the usual correction for sag is based upon the assumption that the supports are at the same height, it will be shown that if the supports differ in level by an amount h, the usual catenary approximation will apply when an independent correction is made for slope, the total correction in the case of a single sag being*

$$-\left\{\frac{l}{24}\left(\frac{wl}{p}\right)^2 + \frac{h^2}{2l}\right\} \qquad \text{(see p. 349).}$$

Low coefficient tapes. The most that the surveyor can say is that a thermometer measures its own temperature, in regard to the relevant corrections in the field. However, the introduction of special steels such as invar, " Konstat ", and " Permant " has done much to overcome these vagaries in base-line measurement, making it possible to measure precise bases without the complicated bars of early practice.

The researches on the properties of nickel steel alloys in 1906 by Dr. Guillaume, of the French Bureau of Weights and Measures, led to the discovery of invar, the least expansible alloy, which contains about 36 per cent of nickel. Its coefficient α, which is possibly the lowest of known metals or alloys, varies in different specimens, not only with the percentage of nickel, but also with the temperature, and is further influenced by heat and mechanical treatment and the contained proportions of carbon, silicon, and manganese. Since the value of α rarely exceeds $0{\cdot}5 \times 10^{-6}$ per degree F., the material is especially suitable for base tapes, obviating the uncertain and usually crude corrections for temperature under average conditions. The tensile strength of invar varies from 100,000 to 125,000 lb. per sq. in., with an average elastic modulus E of 22×10^6 lb. per sq. in., the elastic limit occurring at about 75 per cent of the ultimate strength.

Experiment has failed to detect permanent set under loads of an eighth the ultimate strength. But the alloy is much softer than ordinary steel, and is so readily bent that tapes should be handled with great care, while *set* should be avoided by using winding drums of ample diameter. The tapes should be cleaned and oiled after use, even though their resistance to

corrosion is much greater than that of bright steel. Care should be taken that the oil or grease is free from acid.

Invar and other brands of tapes can be obtained in lengths of 100 ft. to 300 ft. in the $\frac{1}{4}$ in. width, and in the 6 mm. width between 24 and 30 to 100 metres. Their cost prohibits them from ordinary use.

Jaderin's method. The economy of the conjoint use of steel and brass wires has never been rightly appreciated, apart from the check afforded in the duplicate measurement. As regards temperature only, the method may be summarised concisely as follows for measuring heads adjustable to end graduations.

(1) Measure the standard base of l ft. with the steel and brass wires, the fixed lengths being respectively l_1 and l_2 at an assumed common temperature t.

(2) Measure the length of the base line with the steel and brass wires, the observed lengths being L_1 and L_2 respectively at an assumed average temperature T.

(3) Assuming that t_0 is the "normal" temperature, namely, that at which the wires have the same length, and that l_0 is the nominal length of these, determine the temperature rise above normal at standardisation; namely,

$$t - t_0 = \frac{l_1 - l_2}{(\beta - \alpha) l_0},$$

α and β being the coefficients of expansion of steel and brass respectively.

(4) Similarly compute the temperature rise above normal for field measurement, which is likewise approximately

$$T - t_0 = \frac{L_1 - L_2}{(\beta - \alpha) L_1}.$$

(5) Eliminate the normal temperature t_0 from these equations, and so determine the difference θ between field measurement and standardisation:

$$T - t = \frac{1}{\beta - \alpha}\left\{\frac{L_1 - L_2}{L_1} - \frac{l_1 - l_2}{l_0}\right\} = \theta.$$

(6) Calculate the length of the base line corrected for temperature with the steel and brass wires respectively by the formulae

$$\frac{l_1}{l_1 \pm k_1} L_1 \quad \text{and} \quad \frac{l_2}{l_2 \pm k_2} L_2,$$

where $k_1 = \alpha\theta l_0$ and $k_2 = \beta\theta l_0$ accordingly, or $\alpha\theta l_1$ and $\beta\theta l_2$ sufficiently near.

The process is modified if adjustable vernier sleeves are used.

When the vernier is used from a fixed mark on the measuring head, "readings" not "lengths" are observed, the lower temperature giving a *higher reading* and a *smaller length*. Thus if θ algebraically expresses the difference between field and standard temperatures:

Reading at field temperature $= l_1(1-\alpha\theta)$ and $l_2(1-\beta\theta)$.

The base has been measured in lengths giving this reading ; hence in a base of length L_1 there are $n = \dfrac{L_1}{l_1(1-\alpha\theta)}$ lengths, which are each l ft. at standard temperature ; namely, $\dfrac{lL_1}{l_1(1-\alpha\theta)}$.

Colby compensated bars. This apparatus, as described by Bourns, Frome, and others, was used by the Ordnance Survey in measuring the Lough Foyle base in 1826, a length of 8 miles being laid down, and extended to 10 miles by triangulation.

Fig. 146 shows diagrammatically a plan of the bars, which were of brass $\frac{5}{8}$ in. wide, $1\frac{1}{2}$ in. deep, and nominally 10 ft. in length, being riveted together at the centre, $1\frac{1}{2}$ in. apart. The metals were selected so as to obtain a ratio of expansion of 3 to 5 ; and by means of a lever

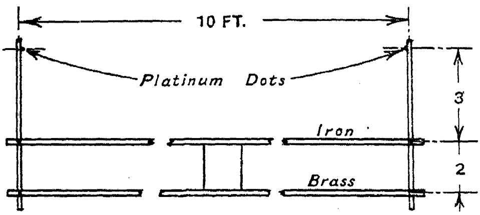


FIG. 146.

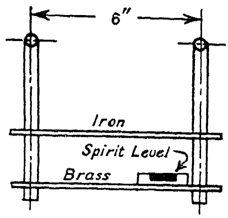


FIG. 147.

system of tongues on pin pivots, a constant length of 10 ft. was read at platinum dots through compensated microscopes. The brass bars were coated with a non-conducting substance to overcome the susceptibility to changes in temperature.

The compensated microscopes, shown in plan in Fig. 147, were similarly constructed with bars $\frac{1}{2}$ in. broad, $\frac{3}{8}$ in. thick, spaced $2\frac{1}{2}$ in. apart, extension levers carrying the microscopes. Levelling with the aid of an attached level was effected by means of a tribach head, lateral adjustment being also provided.

The outfit consisted of coffers in sets of five with bars and microscopes, the coffers resting on trestles so as to give a 52-ft. length of base. The rate of measurement was about one mile in twenty-one days. Improved models with light tubular casings superseded the coffers in recent work ; and in India a mile was measured in five days, the probable error of a single measurement being about $\pm 1{\cdot}5 \times 10^{-6}$.

Auxiliary operations. The following operations may arise in connection with base-line measurement : (i) running a broken base, (ii) interpolating a portion of base ; and (iii) extending a base.

(i) **Broken base.** Frequently some obstruction intervenes, and it is necessary to break the base AB into two sections AC and CB, the deflec-

tion or exterior angle at C being 3°–5° desirably. The lengths $AC=b$ and $CB=a$ are measured in the usual way, and AB is calculated as follows, γ being expressed in minutes (Fig. 148).

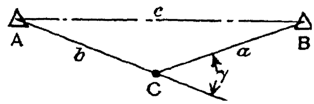


FIG. 148.

$$AB=a+b-\frac{ab\gamma^2}{a+b}4{\cdot}2308\times10^{-8},$$

which follows from the "cosine rule" after expanding $\cos\gamma$ and then expanding by the binomial theorem, the numerical multiplier being $\frac{1}{2}(\sin 1')^2$.

The sine rule should be used when γ necessarily exceeds 5°.

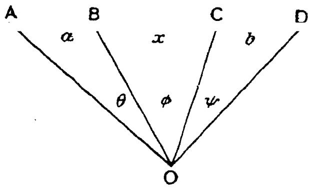


FIG. 149.

(ii) **Interpolation.** Sometimes it happens that an otherwise desirable base contains a portion BC which cannot be measured directly, although alignment is in no way impeded (Fig. 149). In this contingency, $AB=a$ and $CD=b$ are measured, and the angles θ, ϕ, and ψ, subtended by the portions of the base, are observed with a theodolite at O. The problem is then one in elementary trigonometry, and the unmeasured portion may be calculated from

$$x=-\tfrac{1}{2}(a+b)+\sqrt{\frac{ab\sin(\theta+\phi)\sin(\phi+\psi)}{\sin\theta\sin\psi}\left(\frac{a-b}{2}\right)^2}.$$

(iii) **Extension of base.** Very often a base line is much shorter than the average length of side in the triangulation net, and it must be extended through the medium of well-conditioned triangles until its length approximates to the average side of the system. Also the end stations may be inconveniently placed with respect to the triangulation stations, and extension by triangulation may afford a means of overcoming this difficulty.

The process is indicated wholly or in part in Fig. 150, where in conjunction with precise angular measurement the angles are developed with the 60° ideal, restricting the limits to 30° and 120°.

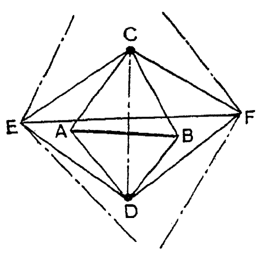


FIG. 150.

Reduction to sea-level. The final correction of a base line consists in reducing its length to sea-level datum, as shown in Fig. 151, where L is the length at average height h and l is its sea-level reduction.

Thus, $$\frac{l}{L}=\frac{R}{R+h}=\frac{1}{1+h/R}=\left(1+\frac{h}{R}\right)^{-1},$$

and since h/R is a very small fraction,

$$l=L\left(1-\frac{h}{R}\right).$$

Strictly, however, a base consists of a number of sections forming a continuous polygonal outline which approximates to the earth's mean curvature. Each section is comprised of several lengths of tape or bar reduced to the horizontal determined by the line of sight at the several settings up of the level, sights involving curvature correction not being normally involved.

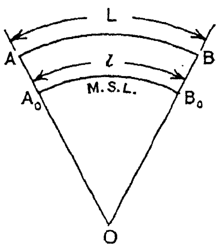


FIG. 151.

Now it follows from the foregoing expression that if in the measurement of a base line there are n sections of length λ with average heights in section of h_1, h_2 ... h_n above mean sea-level, the radial projection at M.S.L. is

$$l=n\lambda-\frac{\lambda}{R}(h_1+h_2+\ldots h_n)\frac{n}{n}=L-\frac{L}{R}\Sigma\frac{h}{n},$$

since $L=n\lambda$.

Thus the correction to the observed length L is $-\frac{L}{R}$ (average height of base h_0), h_0 being the mean elevation for the several sections.

Conclusion. The triangulation net or system having been established, the next operations consist in making precise observations of the horizontal and vertical angles : subjects treated respectively in Arts. 3 and 4 (pp. 353, 359). Also one or more bases of verification may be measured, in order to check their computed lengths. Upon the completion of the triangulation, the angles will be adjusted in accordance with the methods of Sect. VI (pp. 418–422), spherical excess will be calculated, and finally the sides will be computed by one or other of the methods of Art. 7 (p. 389). Normally, when the sides do not exceed 10–15 miles, the triangles will be regarded as plane.

†. A broken base line PQ, about $1\frac{1}{2}$ miles in length, was measured with an invar tape, nominally 300 ft. in length and weighing 5·040 lb., the cross-sectional area being 0·005 sq. in.

(A test report certifies this tape to be 299·9730 ft. under 10 lb. pull on a plane surface, and also states that Young's modulus for the material is 22×10^6 lb. per sq. in.)

During measurement the tape was suspended in three equal sags for 27 lengths over intermediate tripods, the end tripods being fitted with micrometer heads adjustable to the end graduations. In the field a tension of 15 lb. was applied by equal suspended weights at each end, the tension being transmitted by means of wires in the grooves of pulleys.

The base was necessarily deflected from the line proper at the end of the 11th length, and *nine* measurements were made thus, when, on a deflection of 1° 18′ 15″, Station Q was reached with a further *seven* lengths and a *fraction*, the terminal fraction $(27-Q)$ being 84·624 ft. for a single sag under 15 lb. pull.

Station P was 848·80 ft. above sea-level datum and the observed differences of elevation between the measuring tripods in the 28 measurements were in order :

0·12, 0·36, 0, −0·24, 0·48, −0·12, 0, −0·24, −0·60,
0·12, 0·24, 0·36, 0·24, 0·12, 0, 0·24, 0·36, −0·12,
0, −0·48, −0·24, −0·12, 0, −0·36, 0, −0·12, 0, −0·24.

Calculate *accurately* the length of the base at sea-level assuming the earth's mean radius to be 3970 miles. (U.L.)

Increased length under 15 lb. pull $= \dfrac{5 \times 299{\cdot}973 \times 12}{0{\cdot}005 \times 22 \times 10^6 \times 12} = 0{\cdot}0136$ ft.

Standard length $= 299{\cdot}9730$

Field length $= 299{\cdot}9866$,,

Sag correction (300 ft.) $\quad = -\dfrac{300\,(5{\cdot}04)^2}{24\,(3 \times 15)^2} = -0{\cdot}1568$ ft.

for terminal length, $x = -0{\cdot}0317$ ft.

Level corrections: $-\dfrac{h^2}{2l}$, where l is the sloping distance between the tripods $= -0{\cdot}24 \times 10^{-4}k^2$, where k is the number of 0·12 units, or $0{\cdot}85k^2 \times 10^{-4}$ for the terminal portion.

Broken base : $\quad 2(a+b)$

$$\text{Correction} = -\frac{2183{\cdot}4 \times 2698{\cdot}47 \times (78{\cdot}25)^2}{9763{\cdot}73}(0{\cdot}0002909)^2 = -0{\cdot}2483 \text{ ft.}$$

Segment	Obs. length (ft.)	Sag corr.	k^2	Level corr.	Corrected lengths
$P-11$	3299·8526	1·7348	65	0·0016	3298·1262
$11-20$	2699·8794	1·4112	44	0·0011	2698·4671 (b)
$20-Q$	2184·5302	1·1293	15	0·0007	2183·4002 (a)

Corrected total	8179·9935 ft.
Broken base	− 0·2483
Corrected length at elev. 848·80	8179·7452 ft.

Correction to sea-level $= -\dfrac{L\Sigma h}{Rn} = -\dfrac{L}{R}$ (average height of base).

Average height is 848·8 ft. since sum of level differences is zero

$$\frac{8179{\cdot}7452 \times 848{\cdot}8}{3970 \times 5280}$$

Reduced length of base $= 8179{\cdot}4140$ ft.

Example†. Jaderin's method was employed in measuring a base line with steel and brass wires nominally 25 metres in length. Before measurement the wires were tested (under field tension) on a standard base of 25·0024 m. and the standard was observed to be 24·9936 m. on the steel wire and 24·9842 m. on the brass wire, the centigrade coefficients of expansion being $1{\cdot}10 \times 10^{-5}$ and $1{\cdot}90 \times 10^{-5}$ respectively.

The observed lengths of the base were 1278·7240 m. and 1278·7106 m. with the steel and brass wires respectively.

Determine the correction for temperature to be applied to these base line measurements, verifying the results.

State concisely what other corrections will be involved in the measurement of a base by the above method. (U.L.)

Using the notation of p. 341 :

$$l_1 - l_2 \qquad 24{\cdot}9936 - 24{\cdot}9842$$

$$T - t_0 = \frac{L_1 - L_2}{L_1(\beta - \alpha)} = \frac{1278{\cdot}7240 - 1278{\cdot}7106}{1278{\cdot}7240\,(0{\cdot}8 \times 10^{-5})} = 1{\cdot}31.$$

Difference between field temperature and standard temperature, $T - t = \theta = 45{\cdot}69^\circ$ C., which is negative since field temperature is below standard temperature. (Note distinction between "reading" and "length", p. 341.)

Steel reading in field $= l_1 + \alpha\theta l_0 \quad = 24{\cdot}9936 + 45{\cdot}69 \times 1{\cdot}1 \times 10^{-5} \times 25$
$= 25{\cdot}0062$ m.

True length of base by steel wire $= \dfrac{25{\cdot}0024 \times 1278{\cdot}7240}{25{\cdot}0062} = 1278{\cdot}5302$ m.

Brass reading in field $= l_2 + \beta\theta l_0 \quad = 24{\cdot}9842 + 45{\cdot}69 \times 1{\cdot}9 \times 10^{-5} \times 25$
$= 25{\cdot}0059$ m.

True length of base by brass wire $= \dfrac{25{\cdot}0024 \times 1278{\cdot}7106}{25{\cdot}0059} = 1278{\cdot}5302$ m.

Example†. The following data refer to the conjoint use of a steel and a brass wire in the measurement of a base line, the wires having a nominal length of 200 ft.

B.S.G. 21. Steel, 0·427 lb./200 ft.; coeff. expansion, $6{\cdot}2 \times 10^{-6}$ per 1° F.

B.S.G. 20. Brass, 0·572 lb./200 ft.; coeff. expansion, $10{\cdot}4 \times 10^{-6}$ per 1° F.

The respective standard pulls were also used in the field, that on the brass being fixed at 5 lb., while the pull on the steel wire was so calculated that the sagging effect of the single catenary was exactly the same in each case.

The lengths employed were fixed by adjustable nipples to the length of a standard base of 200·014 ft. at a temperature of 50° F., and in the field the base measured 5755·458 ft. and 5754·177 ft. with the steel and brass wires respectively.

Determine the length of the base reduced for a standard temperature of 62° F.

State also what further corrections must be made. (U.L.)

Here the normal temperature t_0 and the test temperature are the same, and the difference between the field and test temperatures will be

$$\theta = \frac{L_1 - L_2}{L_1(\beta - \alpha)} = \frac{5755{\cdot}458 - 5754{\cdot}177}{5755{\cdot}458\,(0{\cdot}42 \times 10^{-5})} = 52{\cdot}993^\circ \text{ F.};$$

or 103° F., which is 41° above standard of 62° F.

Steel length at standard temp. $= l_1(1 + \alpha t) = 200{\cdot}014\,(1 + 12 \times 6{\cdot}2 \times 10^{-6})$.
$= 200{\cdot}029$ ft.

Brass ,, ,, ,, $= l_2(1 + \beta t) = 200{\cdot}014\,(1 + 12 \times 10{\cdot}4 \times 10^{-6})$.
$= 200{\cdot}039$ ft.

Steel length in field $= l_1(1 + \alpha\theta) = 200{\cdot}014\,(1 + 53 \times 6{\cdot}2 \times 10^{-6})$
$= 200{\cdot}080$ ft.

Brass ,, ,, $= l_2(1 + \beta\theta) = 200{\cdot}014\,(1 + 53 \times 10{\cdot}4 \times 10^{-6})$
$= 200\ {\cdot}124$ ft.

Lengths at standard temperature :

By steel wire, $\dfrac{200{\cdot}029}{200{\cdot}080} \times 5755{\cdot}458 = 5753{\cdot}991$ ft.

By brass wire, $\dfrac{200{\cdot}039}{200{\cdot}124} \times 5754{\cdot}177 = 5751{\cdot}733$ ft.

Sag correction per length for both wires,

$$x = -\frac{}{24} \qquad = -\frac{l}{24}\left(\frac{w_2}{p}\right)^2 = -0{\cdot}10906 \text{ ft.},$$

giving a total correction of $-3{\cdot}059$ ft. ($p_1 = 3{\cdot}73$ lb.)

The length thus corrected is to be reduced to horizon and sea-level, though, apart from the effects of an abnormally wide range of temperature, the mean value could not be accepted, even in lower grade work.

Example†. In measuring a series of short base lines with a 100 ft. steel tape, the use of sag corrections was obviated by employing the "normal" tension, or the pull under which the stretch of the tape neutralises its sagging effects. Nominal lengths of 100 feet, suspended in three equal sags, were used consistently, the length employed having been calculated to be 100 ft. precisely under *no* pull on a plane surface at a standard temperature of 62° F.

Tests with a similar tape showed a value of 30×10^6 per sq. in. for Young's modulus, the weight per ft. run being 0·0125 lb., and the cross-sectional area 0·004 sq. in.

Experiments with the steel balance employed showed that the probable error of reading to $\frac{1}{4}$ lb. was 0·04 lb. between 9 and 12 lb.

Determine (*a*) the normal tension per 100 ft. length, and the cumulative error due to reading to the nearest $\frac{1}{4}$ lb. division of the balance, and (*b*) the total error in a base of 25 lengths, assuming that the temperature changes from standard were inappreciable. (U.L.)

Since there is no initial pull p_1, it is unnecessary to avoid a complex cubic by calculating the length l_0 under no pull from $\frac{(p-0)l_1}{AE} = (l - l_0)$, where l is the length under the given pull.

$$\text{Hence} \quad \frac{(p-p_1)}{AE} = \frac{d}{24}\left(\frac{wd}{np}\right)^2 \text{ reduces to } p^3 = \frac{AEw^2 d}{24n^2} = 866{\cdot}85\,;$$

and the normal tension $p = 9{\cdot}5439$ lb.

The cumulative error δp in reading to the nearest $\frac{1}{4}$ lb. is only $-0{\cdot}0439$ lb., while the probable error per reading is $\pm 0{\cdot}04$ lb.

The following solution is sufficiently accurate :

Cumulative error.

$$\text{By stretch, } \delta e = \frac{l}{AE} \cdot \delta p = -\frac{100 \times 12 \times 0{\cdot}0439}{0{\cdot}004 \times 30 \times 10^6 \times 12} = -0{\cdot}0000366 \text{ ft.}$$

$$\text{By sag, } x, \quad \delta x = -\frac{w^2 d^3}{24n^2} \times \left(\frac{2}{-p^3}\right)\delta p = -\frac{(0{\cdot}0125)^2 \times (100)^3 \times 2 \times 0{\cdot}0439}{24 \times 9 \times (9{\cdot}5439)^2}$$
$$= -0{\cdot}00007308 \text{ ft.}$$

$$\text{Total cumulative error per length} = -(0{\cdot}0000366 + 0{\cdot}00007308)$$
$$= -0{\cdot}0010968 \text{ ft.}$$

$$\text{Total cumulative error in 25 lengths} = -25 \times 0{\cdot}00010968$$
$$= -0{\cdot}002742 \text{ ft.}$$

Probable error.

$$\delta e = \pm\frac{0\cdot04}{0\cdot0439}\times 0\cdot0000366 = \pm 0\cdot0000333 \text{ ft.}$$

$$\pm\frac{0\cdot04}{0\cdot0439}\times 0\cdot00007308 = \pm 0\cdot00006659 \text{ ft.}$$

Total probable error per length

$$= \pm 0\cdot0000333 \pm 0\cdot00006659 = \pm 0\cdot00009989 \text{ ft.}$$

Total probable error in 25 lengths

$$= \pm\sqrt{25}\times 0\cdot00009989 = \pm 0\cdot00049945 \text{ ft.}$$

Total error : $-0\cdot002742 \pm 00049945 = -0\cdot0022426$ or $-0\cdot0032415$ ft.

Example†. Derive an expression to be made for the effects of sag and slope in base line measurement, introducing the case where the tape or wire is supported at equidistant points between measuring pegs or tripods.

The curve assumed by a presumably inelastic tape will be a very flat catenary, which approximates closely to a parabola in which s, the length of the tape, is very nearly equal to the horizontal distance l between supports at the *same level* when a pull of p lb. is applied, the tape weighing w lb. per ft.

$$s = \frac{2p}{w}\sinh\frac{l}{2p/w} = \frac{p}{w}\left(e^{wl/2p} - e^{-wl/2p}\right)$$

$$= \frac{2p}{w}\left\{\frac{wl}{2p} + \frac{1}{3\,!}\left(\frac{wl}{2p}\right)^2 + \frac{1}{5\,!}\left(\frac{wl}{2p}\right)^5 \dots\right\} = l + \frac{1}{24}\left(\frac{w}{p}\right)^2 l^3,$$

on neglecting the third and following terms.

Then the sag correction is

$$s - l = \frac{1}{24}\left(\frac{w}{p}\right)^2 l^3 = \frac{l}{24}\left(\frac{wl}{p}\right)^2. \quad \dots\dots\dots\dots\dots\dots\dots\dots\text{(I)}$$

If there be n intermediate supports, each a distance l apart horizontally in a total distance d, $nl = d$, and

$$\text{total sag effect } x_n = \frac{w}{24}\left(\frac{wa}{np}\right.$$

In actual measurement the supports are seldom at the same level, but the tension p is horizontal at the lowest point. Thus, the span l is divided into two parts l_a and l_b, as in Fig. 152, where the origin is assumed at O, the respective sags being a and b.

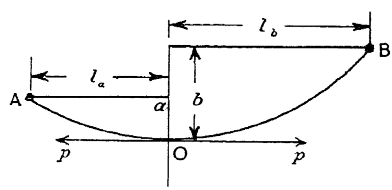


Fig. 152.

It follows, therefore, that there are two parabolas,

$y=\frac{ax^2}{l_a^2}$ and $y=\frac{bx^2}{l_b^2}$ with slopes $\frac{dy}{dx}=-\frac{2ax}{l_a^2}$ and $+\frac{2bx}{l_b^2}$ respectively.

Thus, the length of the curve

$$s=s_1+s_2=\int_0^{-l_a}-\left[1+\frac{2a^2x^2}{l_a^4}\right]dx+\int_0^{+l_b}\left[1+\frac{2b^2x^2}{l_b^4}\right]dx$$

$$=[l_a+l_b+\tfrac{2}{3}(a^2/l_a+b^2/l_b)]. \qquad \text{.............................(1)}$$

From the statics of the figure,

$$p=\frac{wl_a^2}{2a}=\frac{wl_b^2}{2b}\ ; \quad \text{and } a/l_a^2=b/l_b^2\ ; \qquad \text{....................(2)}$$

and on substituting these values in (1),

$$s-l=\frac{1}{6}\frac{w^2}{p^2}(l_a^3+l_b^3). \qquad \text{......................}$$

By writing $\frac{1}{2}l-x=l_a$ and $\frac{1}{2}l+x=l_b$,

$$s-l=(\text{sag}+\text{level})\text{ correction}=\frac{w^2}{6p^2}\{(\tfrac{1}{2}l+x)^3+(\tfrac{1}{2}l-x)^3\}$$

$$=\frac{w^2}{6p^2}\left\{\frac{l^3}{4}+\tfrac{3}{4}l(l_b-l_a)^2\right\}=\frac{w^2}{24p^2}l^3+\frac{w^2}{8p^2}\frac{(l_b^2-l_a^2)}{(l_a+l_b)}\ \ldots.$$

On substituting for l_b and l_a from (2) in the last term of (3), this term reduces to

$$\frac{(b-a)^2}{2(l_b+l_a)}=\frac{h^2}{2l}, \qquad \text{...............................(III)}$$

where h is the difference of level of A and B.

Thus thè total correction is the sum of the separate corrections for sag and slope.

Effects of latitude and altitude in base line measurement. When a tape or wire measurement is made with suspended weights in a latitude different from that of the place of standardisation, it may be necessary in more precise work to make some allowance for the change in g, the acceleration due to gravity. The tension exerted by the weights and the weight of the tape are proportional to g, and, although the form of the catenary remains constant, the tape undergoes a small deformation by the change from standard tension.

Among the various formulae which express the variations of g, one suggested by Helmert may be embodied, with the corrections for height above mean sea-level and mass between the latter and the station, $-\frac{2hg}{R}$ and $+\frac{3}{4}\frac{hg}{R}$ respectively, leading to the expression :

$$g=g_0\frac{(1+0{\cdot}00531\sin^2\lambda)}{\left(-\frac{2h}{R}+\frac{3}{4}\frac{h}{R}\right)^{-1}},$$

where g_0 is the acceleration at sea-level on the equator, g is the corresponding value at an elevation h in latitude λ, and R is the mean radius of the earth, 21×10^6 ft. approximately.

Thus if g_1 and g_2 are the respective values at the stations of standardisation and measurement, both computed in terms of g_0 by the above formula, and if dp is the change produced in the nominal tension p, $dp = \frac{g_2 - g_1}{g_1} p$, which will lead to a correction in the stretch that will be plus or minus according as $g_2 \gtrless g_1$.

If, however, a spring balance is used, the pull is unaffected by g, but the weight of the tape is altered, producing a variation in the sag effect.

QUESTIONS ON ARTICLE 2

1†. Describe the process of measuring a base line with the steel tape in a large topographical survey, the limit of error of measurement not exceeding 1 in 300,000. A 300 ft. invar base-tape was standardised under a pull of 15 lb., the tension under which it was to be used in the field. During operations the spring balance broke, and was replaced by one reading only to 10 lb. The base was measured with two intermediate supporting pegs, dividing the tape length into three equal sags.

The nominal total length of the base was 5,940 ft., eight full length measurements being made with the original spring balance, eleven with the smaller one, and, with this latter, the terminal length of 240 ft. State the total allowance to be made in the sag correction, given that the latter was originally calculated to be 0·14 inch per tape length. What other allowance would have to be made? Using reasonable dimensions, indicate its approximate value per tape length. (U.L.)

[Total sag correction 0·3955 ft. against 0·2277 ft. with original balance. Correction for elastic stretch under pull less than standard, $-0{\cdot}00667$ ft. per tape length with breadth $\frac{1}{4}$ in. and thickness 0·03 in., E being 30×10^6 lb./sq. in.]

2†. A steel tape was exactly 100 ft. in length on a plane surface under a pull of 10 lb. at the standard temperature of 62° F. Its cross-sectional area was 0·008 sq. in. and its weight 2·80 lb., the coefficient of expansion being $6{\cdot}5 \times 10^{-6}$ per degree F. and the modulus of elasticity 30×10^6 lb. per sq. in.

A straight base line was measured with this tape suspended in twenty-four spans under a pull of 20 lb., the temperature being 72° F. for the first six lengths, 80° F. for the next ten, and 84° F. for the remainder. The first eight lengths were on the level, the next eight on a uniform slope of 1 in 125, and the final eight were again on the level, the average elevation being 1320·0 ft. above sea-level datum.

Compute the true length of the base at sea-level, assuming the earth's mean radius to be 4000 miles, and working to three decimal places.

(U.L., Cart.)

[Length, 2398·234 ft.]

3†. Describe with reference to sketches Bessell's Zinc and Iron base-bars in the (*a*) compensating and (*b*) non-compensating patterns, explaining the use of the glass wedges in the latter apparatus.

4†. Steel and brass wires, nominally 300 ft. in length, were tested under field tension on a standard base of 300·018 ft., which latter was observed to be 299·956 ft. on the steel wire and 299·924 ft. on the brass wire, the coefficients of linear expansion of the material being respectively $6{\cdot}5 \times 10^{-6}$ and $10{\cdot}6 \times 10^{-6}$ per degree F. The lengths of the base corrected for sag were 3464·720 ft. and 3464·525 ft. when measured with the steel and brass wires respectively. Determine the length of the base line, corrected also for temperature, verifying the result with respect to both wires.

[Length : by steel, 3465·133 ft. ; by brass, 3465·126 ft.]

5†. A base line was measured with an invar tape suspended in three equal sags under a field tension of 20 lb. applied by means of (*a*) weights at each end, and (*b*) a precise spring balance, the lengths duly corrected for sag and slope being respectively 125529·963 ft. and 125521·212 ft. at an elevation of 124 ft. in latitude 26° 30′ N.

The tape was standardised as exactly 300 ft. under field tension at London in latitude 51° 25′ N. at an elevation of 56 ft., the observed elongation being 0·001818 ft. per lb. for the tape length.

Given that the tape weighed 7·5 lb. in London, and that its cross-sectional area was 0·0075 sq. in., determine the correction due to gravity in each case, and state the mean length of the base reduced to sea-level.

N.B. The earth's mean radius may be assumed to be 21×10^6 ft. (U.L.)

[(*a*) Sag unaltered, modified total stretch, $-0{\cdot}0332$ ft. (*b*) Tape lighter, modified total sag effect, $+0{\cdot}3565$ ft. 125,529·508 ft.]

6†. Show that in base line measurement with tapes and wires in flat catenary with the supports at different levels, the total correction will be $-(x+c)$, where x is the parabolic approximation for sag between level supports and c, the level or slope, correction, taken permissibly to the first approximation. (U.L.)

7†. A base line *AB* was measured in difficult country, and was necessarily laid down in two straight sections, *AC*, *CB*, the exterior angle at *C* being 2° 50′ 10″. The corrected horizontal lengths of *AC* and *CB* were 10842 ft. and 12,646 ft. respectively.

(*a*) Calculate the length of the " broken " base *AB*. [23480·849 ft.]

When the base line apparatus was not available, it was found that *B* was unsuitably placed with regard to the triangulation stations, and it was decided to extend it a further 4500 ft. to *D*.

(*b*) Indicate on a neat sketch how this extension should be carried out.

The average elevation above mean sea-level was 874 ft. between *A* and *C* and 932 ft. between *C* and *B*, while an average height of 956 ft. was observed between *C* and *D*.

(*c*) Determine the length of the extended base, reduced to mean sea-level, assuming the earth's mean radius to be 3960 miles. [27,979·613 ft.]

8. Sketch neatly one type of apparatus used to measure length accurately, using tapes or wires suspended in catenary. Explain the procedure followed, and the corrections which have to be applied. (I.C.E.)

9. Give reasons for the modern preference for using tapes or wires rather than rigid bars in the measurement of primary base lines.

Describe the apparatus required for the measurement of a base line of geodolic standard by means of an invar tape or wire supported at its ends only. (I.C.E.)

10. A base line for a triangulation is to be measured with a steel tape. Give a complete list of the necessary apparatus with sketches and describe how you would carry out the measurement. Give approximate dimensions of the tape you would use. What kind of steel should it be made of? Give your reasons. Write down a complete list of corrections which must be applied to the measured length, indicating whether these corrections are additional or subtractive. (I.C.E.)

11. You measure a base of six bays with a 100 feet steel tape in catenary at a temperature of 94° F.

The supports of the tape are 1·65 ; 2·43 ; 2·64 ; 4·08 ; 3·21 and 3·94 feet above the first support.

The steel tape in catenary with supports at the same level measures 100·073 feet at 70° F. Co-efficient of expansion 0·0000065 per 1° F.

What is the true length of the base? (T.C.C.E.)

[600·500 ft.]

12. Show that the correction to a length measured with a tape held in catenary, and with the ends at the same elevation is $\frac{w^2l^3}{24T^2}$,

where w is the weight of 1 ft. of the tape,
l is the length of the catenary,
T is the tension applied.

A tape held in catenary at a tension of 20 pounds at a temperature of 85° F. is compared with a standard base and is found to be 100·037 feet.

What is the true length of the tape held in contact with a flat horizontal surface under a tension of 15 pounds at a temperature of 60° F.?

$w = 0{\cdot}01$ pounds.

Coefficient of expansion 0·0000062.

„ tension 0·0000083. (U.C.T.)

[100·046 ft., *tape reading*.]

ARTICLE 3 : PRECISE ANGULAR OBSERVATIONS

There are two general methods of observing horizontal angles with great accuracy : (I) method of series ; (II) method of repetitions, both of which comprehend various modifications, being adapted to eliminate such errors as affect the desired degree of accuracy of the survey.

METHOD OF SERIES

This method, which is that adopted by the Ordnance Survey, involves the measurement of the angles several times, the mean of the observations being taken as the most probable values. The synonymous use of the term "reiteration" is avoided for obvious reasons.

O.S. methods of observing horizontal angles. Let 1, 2, 3, etc. (Fig. 153), represent stations to be observed in order of azimuth with a theodolite at A. The common vertex Station 1 is first sighted, and the microscopes read and recorded. Station 2 is then observed likewise; then 3, and other stations, if any; coming round by continuing motion until 1 is again sighted, the agreement of the second and first observations on that station being a test of the stability of the instrument. In observing this round of angles, or "arc", as it is termed, the time interval between the first and second observations of 1 should be as small as possible, consistent with careful observation. Before taking the next arc, the horizontal circle is turned through a certain angle 20° or 30°. This step tends to eliminate the errors of graduation, the readings being taken on a different set of divisions for each arc.

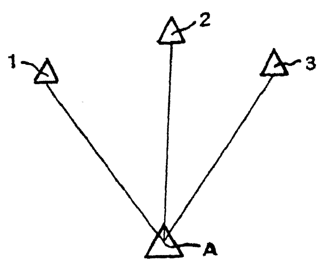


FIG. 153.

Each "arc" at a station should have a "referring point" to which all the angular measurements are referred, the observations of each arc commencing and finishing on this point. Usually a point among those under observation serves the purpose of the referring object. The referring object used on mountain tops consists of two vertical plates, fixed with their edges parallel, at such a distance apart that the light of the sky seen through appears as a vertical line about ten seconds in width.

Larger instruments formerly characterised measurement by series, though to-day the circles seldom exceed 12 in. in primary triangulation, those with 15 in. circles being made to individual requirements.

Three or four micrometers are provided for reading the horizontal circles, the three reading directly to 1 sec., or $\frac{1}{4}$ sec. by estimation. Two or four similar microscopes are fitted to the vertical circle. Sometimes an eyepiece micrometer is fitted in order that the signal may be bisected by a movable vertical hair, the micrometer reading being added to that of the circle micrometers. A sensitive altitude level and a precise striding level are also provided among other refinements. Some models are fitted with a small vertical telescope for exact centring over the station.

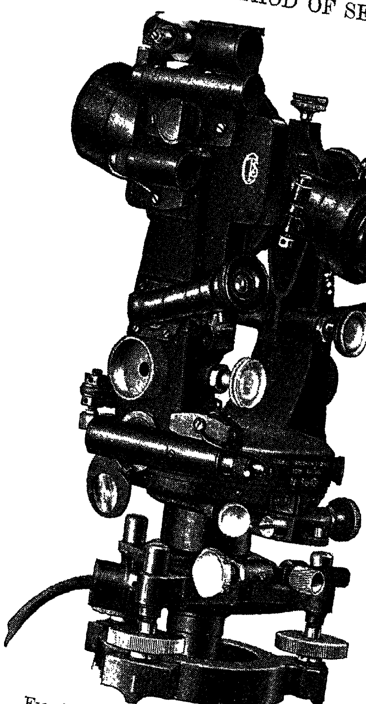


FIG. 154. Geodetic "Tavistock" Theodolite.
(Messrs. Cooke, Troughton & Simms.)

The Geodetic "Tavistock" Theodolite, as used in the re-triangulation of Great Britain (1935), has circles only of $5\frac{1}{4}$ in. and $3\frac{1}{2}$ in. diameter for horizontal and vertical angles respectively.

General precautions in observing angles. (1) **Adjustment of the arc.** When the horizon has been closed, and the first and last readings of the circle do not agree, the error is usually divided among the angles in proportion to their number, irrespective of their magnitude. It is doubtful whether this correction adds to the accuracy of the result, and therefore it is a frequent practice to read the angles without closing the horizon.

(2) **Twist of station.** The upper part of an elevated station is, in clear weather, usually subject to a twisting motion in the direction of the sun's motion. According to experiments, this effect has been observed to amount to 1 sec. of arc per minute of time on a station 75 ft. in height.

The error is eliminated by observing the angles in both directions, first to the right and then to the left—" swing right " and " swing left ". This precaution is also considered to equalise the movements arising from exceedingly small slips of the clamps or " give " at the levelling head or tripod.

(3) **Errors of adjustment.** Errors in horizontal angles due to faulty collimation and horizontal axis are eliminated by using both faces of the instrument—" face left " and " face right ". Errors due to the vertical axis not being truly vertical will not be eliminated, nor, incidentally, errors in vertical angles, except so-called index error.

(4) **Errors of centring.** The errors due to eccentricity are eliminated by reading opposite microscopes.

(5) **Errors of graduation.** The errors arising from irregularities in the graduations of the limbs are eliminated by reading the magnitude of the angles on different parts of the circle. This is effected by advancing the zero after each set of readings through an angle of $360°/Mn$, where M is the number of microscopes read and n the number of sets of observations to be taken.

Precautions against backlash should always be taken in the use of micrometer screws.

Refraction always introduces an element of uncertainty, and lateral refraction is more disconcerting than refraction in altitude. Sights should be kept well above the ground, so far as is practicable. Also observations should be taken in densely-clouded weather, or failing this, from the late afternoon to sunset.

Routine. It is possible only to outline the routine in general terms, the number of sets or combined results being determined by the instrument employed and the degree of accuracy demanded by the order of the triangulation ; secondary, tertiary, etc.

Wherefore, the surveyor should base his procedure upon that of some similar, representative survey, following, for example, the " Instructions " issued by the Mississippi River Commission.

METHOD OF REPETITIONS

In this method the angles are measured separately a number of times, primarily with the view of utilising different parts of the horizontal circle. The process is facilitated by means of the " repeating circle ", usually attributed to Borda, but actually invented by Tobias Mayer in 1752.

The repeating circle theodolite, an instrument expressly designed for this method, was introduced by the French on the triangulation system connecting the observatories of Paris and Greenwich (1783). On the part of the triangulation that fell to England, Ramsden's theodolite (1783)

was used; an instrument of such excellent quality that the repeating circle has never crossed the English Channel. "Repetitions", though used extensively on the Continent and in America, is a method apparently perfect in theory; but in practice, the clamps failing to afford absolute stability, it is liable to repeat the error along with the angle measured, and, in consequence, a small repeating theodolite is inferior to a larger or a modern geodetic theodolite.

General precautions. Reviewing the suggestions given with reference to Series (p. 355): (1) Adjustment of arc is inapplicable; (2), (3) and (4) are precisely the same; while (5) the elimination of errors of graduation is inherent in the method.

Combined result. When all the above precautions are observed, it is necessary to take the angles in "sets", two of which, a clockwise and a counterclockwise set, constitute a "combined result". The mean value of a clockwise set is taken against the mean of a set in the contrary direction, the mean of these two values being taken as that of a combined result.

A good deal of confusion exists in the synonymous use of the terms "series" and "sets". The latter implies a combination of observations either by Series or by Repetitions. Also the term "repetition", when referring to a number of angular measurements, is frequently used in the sense of the term "observation", but, strictly, the first repetition is the second observation; the fifth repetition, the sixth observation.

Procedure. A common practice is to observe the angle to the right six times, reversing the faces of the instrument at the end of the third observation. Then, without resetting the microscope, the explement of the angle is measured, also to the right, the telescope being normal for the first three observations, but inverted for the last three. Theoretically the micrometer reading should come back to the original setting and the sum of the two angles (angle + explement) should be equal to 360°. Any discrepancy, which is the difference between the final reading and the original setting of the microscope, is divided, and one half is assigned as the correction of the angle, the other as the correction of the explement. This method involves all the steps providing the necessary corrections.

Vertical angles. The foregoing methods function only in part in the observation of vertical angles. Face left and right readings utilise only opposite portions of the vertical circle, though the means of the microscope readings eliminate errors of eccentricity. Otherwise, the use of both faces eliminates only the index error; and thus it is essential that the instrument should be accurate in horizontal collimation, for trunnion axis error can be corrected from the readings of the precise striding level and apparent index error likewise from the altitude level. Incidentally, both these levels, if available, should be used to ensure that the vertical axis is truly vertical in all accurate or precise angular measurement.

Example†. A 10-in. theodolite is supplied for a major triangulation, each micrometer microscope reading to 1 sec. and 0·25 sec. by estimation. It is found, however, that the average probable error in a single measurement is 2 sec. when reading the mean of both microscopes. The method of " series " is to be used with a degree of precision equal to that of the base measurements, which are assumed to have a probable error of 1 in 500,000.

Detail concisely the procedure of measuring the horizontal angles at a station to the requisite degree of accuracy, introducing such steps as will eliminate the various errors concomitant with precise angular measurement.

N.B. The method of " repetitions " is unsuitable for use with the instrument supplied. (U.L.)

Eccentricity being eliminated in the mean of the verniers, the remaining errors are treated as follows : (*a*) Adjustment : use both faces ; (*b*) division : use different parts of circle ; (*c*) twist of station (slip of clamps) : take rounds in both directions. Embody these precautions in the procedure.

Since the angles will be measured similarly n times, n is the weight ; and if an error E be made in each single measurement, the error of the average or mean value E_m will be

$$\frac{\sqrt{\text{error square sum}}}{} = \frac{\sqrt{nE^2}}{} = \frac{E}{}$$

Thus if $1/x$ is the specified ratio of precision, 1 in 500,000,

$$\frac{1}{x} = \frac{E}{} ; \frac{1}{206265},$$

reducing E to circular measure.

Hence $103133\sqrt{n} = 500,000$, $\sqrt{\ } = 4{\cdot}86$, and $n = 23{\cdot}7$, or 24 conveniently, as in the following routine :

F.L.	Clockwise ;	Zero at	0°	30°	60°	90°	120°	150°.	Six rounds.
F.R.	Anti- ,,	,,	,,	,,	,,	,,	,,	,,	,,
F.L.	Clockwise	,,	,,	,,	,,	,,	,,	,,	,,
F.R.	Anti- ,,	,,	,,	,,	,,	,,	,,	,,	,,

QUESTIONS ON ARTICLE 3

1†. Draw up a tabular scheme, prescribing the routine of observing the angles of a triangulation net by the method of " Repetitions " with a theodolite in which the probable error is 2 sec. in a single measurement from the mean of both microscopes.

The routine submitted is to anticipate a degree of precision of 1 in 500,000, and must embody such steps as will eliminate the various errors associated with precise angular measurement, the number of repetitions being limited to five in any set of measurements. (U.L.)

2†. Describe methodically how you would observe the horizontal angle of a major triangulation with a 10-inch geodetic theodolite which reads to single seconds ($\frac{1}{4}$ sec. by estimation), a limit of error of 1 in 10^6 being prescribed.

State what errors will be eliminated in the various steps of your procedure. (U.L.)

3†. Explain, giving tabular notes, how you would proceed to observe the angles at one station of a minor triangulation by means of a theodolite on which the microscopes may be read to single seconds.

Add a note giving the reasons of your choice between the methods of Repetitions and Series (Reiteration), and state clearly what errors are eliminated by the various steps in the procedure suggested by your notes. (U.L.)

4†. Draw up a tabular scheme, prescribing the routine of observing the angles in a major triangulation net by the method of "Series" by means of a geodetic theodolite with which the average error has been found to be 1·5″ in a single measurement from the mean of both microscopes.

The routine submitted is to eliminate the various errors associated with precise angular measurement, and the degree of precision is to conform with that of the base measurements, which are assumed to have a probable error of 1 in 10^6. (U.L.)

5†. Draw up the Instructions for the measurement of vertical angles to 1 in 500,000 by means of a 10-inch geodetic theodolite which is provided with a pair of microscopes reading to single seconds ($\frac{1}{4}$ sec. by estimation), the instrument being provided with sensitive striding and altitude bubbles.

Add a note on the effects of atmospheric refraction, and prescribe times of observation accordingly.

6. Give a brief description of the methods of repetition and reiteration as applied to the measurement of horizontal angles with a theodolite. (I.C.E.)

7. Describe various methods of measuring horizontal angles in important triangulation. How are instrumental and observation errors minimised by these methods. (I.C.E.)

8. Name each step in the observer's procedure at a station of minor triangulation. (T.C.C.E.)

ARTICLE 4: TRIGONOMETRICAL LEVELLING

Trigonometrical levelling is the process of determining the differences of elevations of stations from observed vertical angles and known distances, which are assumed to be either horizontal or geodetic lengths at mean sea-level.

The methods are (1) **direct** or (2) **reciprocal**, according as the vertical angles are observed on only one of a pair of stations, or, in order to avoid the uncertainty of refraction, each of the stations is occupied in order that the mean angle may be determined without correction.

The accuracy of direct observations is always influenced by the irregularities in the coefficient of refraction m, while in the reciprocal method the equality of the refraction effects is merely assumed, and for this reason simultaneous observations are desirable, if not always practicable; and, in consequence, the work is often carried out on different days, frequently with impaired precision, at the time of minimum refraction effect.

(1) **Direct observations.** Let A be an instrument station of known elevation above mean sea-level (M.S.L.) and let B be another station, the elevation of which is to be determined, D being the horizontal or the geodetic distance between A and B, and α and θ respectively the vertical angle and the angle at the earth's centre O (Fig. 155).

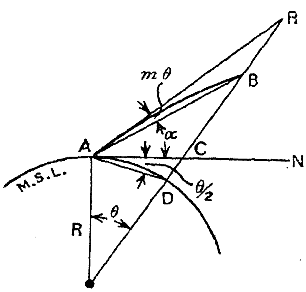


FIG. 155.

Also let B be considered coincident with R prior to taking account of refraction, also omit the "axis-signal" correction, due to the relative heights of the instrument and the distant signal. Two cases arise in practice beyond that in which D is so small in that the effects of curvature and refraction are negligible.

(A) **Great distances.** *When ACB is assumed to be a right angle.* In Fig. 155 it will be seen that the error due to curvature is $CD = D^2/2R$, which with a mean radius R of 20,888,629 ft. is $2{\cdot}3936D^2 \times 10^{-8}$, with D also in feet; or expressed preferably as an angle,

$$\theta/2 = D/2R \text{ radians} = D/2R \sin 1'' = \frac{D}{2 \times 101{\cdot}3} \text{ sec.},$$

or $4{\cdot}9375D \times 10^{-3}$ sec. accordingly, since $R \sin 1'' = 101{\cdot}3$ ft.

The effect of refraction is opposite to that of curvature, tending to increase or decrease the vertical angle α according as α is in elevation $(+)$ or depression $(-)$. It is usually expressed as a coefficient $m(0{\cdot}07$ to $0{\cdot}08)$ of the central angle θ, or $2(CAD)$ giving $0{\cdot}14$ (CAD), and since the latter is a small angle, the effect is very nearly $0{\cdot}14$ of the curvature effect. Hence either $0{\cdot}14$ (CD) or $0{\cdot}14$ (CAD) is commonly deducted from the curvature correction. The resultant correction, $2{\cdot}0106D \times 10^{-8}$ ft. or $4{\cdot}1474D \times 10^{-3}$ sec. with $m = 0{\cdot}08$, is applied in two ways: (a) correcting the height and (b) correcting the angle, though $(1-m)\theta = 9{\cdot}0850D \times 10^{-3}$ is also introduced in the case of very great distances.

(*a*) Correcting the height. $BD = H = D\tan\alpha + \frac{D^2}{2R}(1-2m)$, ...(1)
assuming $AC = AD$.

(*b*) Correcting the angle. $BD = H = D\tan\left(\alpha + (1-2m)\frac{D}{2R\sin 1''}\right)$, ...(2)
with the corrected angle in seconds.

When α is negative, the above change sign in the $(1-2m)$ terms, and if, as in reciprocal levelling, the negative value of α be called β, the $(1-2m)$ term will disappear.

(B) **Very great distances.** *When ACB is taken at its correct value.*

Now in Fig. 155, $BC = D\frac{\sin BAC}{\sin ABC}$, also $ABC = 180° - (AOC + OAB)$;

and $$ABC = 180° - (AOC + 90° + \alpha) = 90° - (\theta + \alpha);$$

whence $$BC = D\frac{\sin\alpha}{\cos(\theta+\alpha)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots .(1a)$$

(*a*) Substituting for θ its value in sec., namely, $D/R\sin 1''$,

$$= \frac{D(\sin\alpha - mD/R\sin 1'')}{\cos\{\alpha + (1-m)D/R\sin 1''\}} + \frac{D^2}{2R}\ldots$$

(*b*) Or correcting the angle from BAC to BAD,

$$\cos(\alpha - m\theta + \theta) \quad - \quad \cos\{\alpha + (1-2m)D/R\sin 1''\} \qquad \ldots(4)$$

$$= D\frac{\sin(\alpha + 4\cdot1474D\times10^{-3})}{\cos(\alpha + 9\cdot0850D\times10^{-3})} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots .(4a)$$

Instrument and signal corrections. So far the differences of elevation have been determined between the axis of the observing instrument and the signal sighted, and at this juncture it is desirable to consider the "axis-signal" correction, introducing the difference in height of the instrument and the signal. Anticipating the Reciprocal Method (p. 363), let H_1 and H_2 be the respective elevations of A and B above mean sea-level, and let h_1 and s_1 and h_2 and s_2 be the corresponding heights of the instruments and signals. These may be applied as (*a*) height corrections or (*b*) angle corrections, though in certain problems it is necessary to correct the angles, applying a correction before calculation.

(*a*) *Height corrections.* These are readily applied after the differences of elevation H have been calculated.

Simple levelling. If h_1 is the height of the axis and s_2 that of the distant signal, $-(s_2 - h_1)$ is to be added algebraically to H, which may be regarded as + and − for angles of elevation and depression respectively.

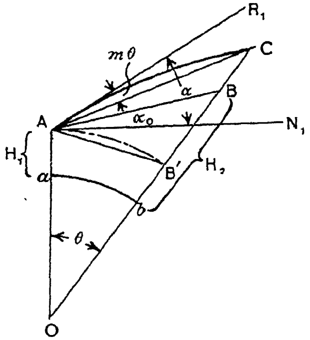


FIG. 156.

Reciprocal levelling. When height corrections can be applied, the apparent difference of elevation is $H_2' - H_1'$, where actually

$$H_2' = H_2 + s_2 - h_1$$

and $\quad H_1' = H_1 + s_1 - h_2$ reciprocally ;

and $\quad H_2' - H_1' = H_2 - H_1 + (s_2 - s_1) + (h_2 - h_1)$

H_2 and H_1 being corrected heights above datum, and not differences of elevation.

(*b*) *Angle corrections.* Consider Fig. 156, where A is the axis of the theodolite, B a point at the same height h_1 above the ground, and C the signal, which is at a height $s_2 - h_1$ above B.

Assuming B' on OC at the same elevation as A above mean sea-level, and AN_1 a horizontal line through A, then α is the vertical angle actually observed, and this embodies the refraction effect represented by the angle $m\theta$ between AR_1 and the curve AC and the axis-signal correction denoted by σ_1, $= BAC$.

Since the effect of refraction upon this small angle σ_1 is not appreciable and θ is always so small that $AB' = D$ very nearly, the obvious correction is

$$\sigma_1'' = \frac{(s_2 - h_1)}{D \sin 1''} \qquad .(5)$$

which is sufficiently accurate for most purposes. It also affords a trial value of σ_1 for the next approximation,

$$\sin \sigma_1 = \frac{(s_2 - h_1)}{D} \cos^2(\alpha + \sigma_1$$

which is derived from the exact relation

$$\sin \sigma_1 = (s_2 - / \quad \frac{\sin ABC}{AC}$$

when $m\theta$ is neglected, and $\frac{1}{2}\theta$ ignored, and AB' taken equal to D.

Simple levelling. Apply the following algebraically, angles of elevation being positive and angles of depression negative :

$$\sigma_1'' = \frac{-2{\cdot}083 \times 10^5 (s_2 - h_1)}{D} \text{ sec., with } D \text{ in feet.}$$

Reciprocal levelling. A similar expression for σ_2'' corresponding to $(s_1 - h_2)$ will occur, the values being subtractive from plus angles and additive to negative angles, following the above algebraical rule.

Reciprocal levelling. Consider the operation shown in Fig. 157, where the angle of elevation α and the angle of depression β have been corrected for heights of axis and signal (or are subject to these, as discussed hereafter). The required difference of elevation $H_2 - H_1$ is found by solving the triangle AOB, the following quantities being given:

$$OA = R + H_1,$$

where R is the earth's mean radius,

$$AOB = \theta = D/R \text{ radians},$$

and $(OAB - OBA)$

$$= \{(90^\circ + \alpha - m\theta) - (90^\circ - \beta - m\theta)\}$$

$$= (\alpha + \beta). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

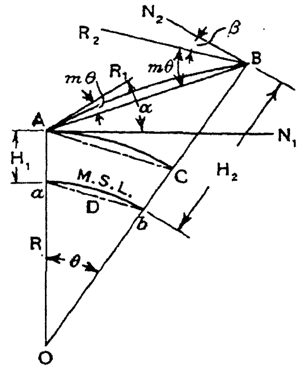


FIG. 157.

Incidentally, $BAC = OCA - OBA$,

while $BAC = BAO - CAO$ and $OCA = CAO$;

whence $2BAC = (\alpha + \beta)$ by adding the values of BAC.

Now $$\tan \tfrac{1}{2}(\alpha + \beta) = \frac{(R + H_2) - (R + H_1)}{(R + H_2) + (R + H_1)} \cot \tfrac{1}{2}\theta,$$

or $$H_2 - H_1 = \tan \tfrac{1}{2}(\alpha + \beta) \tan \tfrac{1}{2}\theta (2R + H_1 + H_2). \quad \ldots\ldots\ldots\ldots (2)$$

As a first approximation, $\tan \frac{1}{2}\theta(2R + H_1 + H_2)$ may be written $D = R\theta$, which is equivalent to taking

$$H_2 - H_1 = BC = D \qquad = D \tan \tfrac{1}{2}(\alpha + \beta). \quad (3)$$

It is evident that this will apply for long distances, for when θ is 1°, D is 364,800 ft. or nearly 70 miles.

For very great distances, expand $\tan \frac{1}{2}\theta$; then

$$H_2 - H_1 = D \tan \tfrac{1}{2}(\alpha + \beta) \left\{1 + \frac{(H_1 + H_2)}{2R} + \frac{D^2}{12R^2}\right\}. \quad \ldots\ldots\ldots (4)$$

In order to apply this last expression, the first approximation of $H_2 - H_1$ in (3) is assumed, and the derived value of H_2 used in the second approximation. In this connection the axis-signal correction should first be applied.

When the difference in elevation is so small with respect to the distance that both α and β are angles of depression, $OAB - OBA$ is now $(\beta - \alpha)$ and

$$H_2 - H_1 = D \tan \tfrac{1}{2}(\beta - \alpha) \left\{1 + \frac{(H_1 + H_2)}{2} + \frac{D^2}{12R^2}\right\}. \quad \ldots\ldots\ldots (4a)$$

(*a*) When height corrections can be applied, the apparent difference of elevation is $H_2' - H_1'$ in the above formulae, where actually

$$H_2' = H_2 + s_2 - h_1 \quad \text{and} \quad H_1' = H_1 + s_1 - h_2 \text{ reciprocally,}$$

or $\qquad H_2 - H_1$

(*b*) When angle corrections are used, these should be applied in the manner described, invariably in determining coefficients of refraction from reciprocal observations.

Coefficient of refraction. The value of the coefficient of refraction may be determined as follows from the reciprocal observations, the angles α and β having been corrected for the heights of axis and signal:

$$N_1AB = (\alpha - m\theta) \text{ and } N_2BA = (\beta + m\theta).$$

Also $\quad N_2BA - N_1AB = \theta$; whence $m = \dfrac{\theta - \beta + \alpha}{2\theta}$,

or
$$m = \frac{1}{2}\left\{1 - \frac{R \sin 1''}{D}(\beta - \alpha)\right\} \ldots\ldots\ldots\ldots\ldots$$
$$= \left(\frac{1}{2} - \frac{50 \cdot 9}{D}(\beta - \alpha)\right),$$

with D in feet and β and α in seconds, and $R = 21 \times 10^6$ ft.

It can be readily deduced from Fig. 157 that the observed angle of depression β always exceeds the observed angle of elevation α, or $\beta = \alpha + \theta(1 - 2m)$, or in seconds, $\beta'' = \alpha + 0{\cdot}00848D$, with D in ft. and $m = 0{\cdot}07$.

Also, α will also be an angle of depression when the arithmetical sum $\beta'' + \alpha''$ is less than $0{\cdot}00848D$ sec.

Example†. Levelling across a wide valley between two trigonometrical stations P and Q was effected by reciprocal theodolite observations on account of the fact that one station was much higher than the other. The calculated distance PQ was 16450 ft., and the heights of the instrument at P and Q were respectively 4·80 ft. and 4·69 ft., being under ground signals 20 ft. in height above each station.

Given that the angle of depression from P was 20° 18′ 15″ and the angle of elevation from Q was 2° 22′ 19″, determine the difference in elevation of the stations, also the coefficient of refraction on the assumption that 1″ at the earth's centre subtends 101·5 ft. at the surface. [log tan 1″ = 6·6855749.] (U.L.)

Reducing the observed angles α and β for axis-signal correction; $\mp$ to $\pm$ angles :

$$\sigma_1 = \frac{(s_2 - h_1)}{D \sin 1''} = \frac{(20 - 4{\cdot}6)}{16450 \tan 1''} = 193{\cdot}10''$$

$$\sigma_2 = \frac{(s_1 - h_2)}{D \sin 1''} = \frac{(20 - 4{\cdot}8)}{16450 \tan 1''} = 190{\cdot}59'' ;$$

leading to the values $\alpha_0 = 2° \, 19' \, 5{\cdot}9''$; $\beta_0 = 2° \, 21' \, 25{\cdot}6$

Difference of elevation,

$$H_2 - H_1 = D \tan \tfrac{1}{2}(\alpha_0 + \beta_0) = 16450 \tan(2° 20' 15{\cdot}75'') = 671{\cdot}59 \text{ ft.}$$

Coefficient of refraction :

$$2m = \left(1 - \frac{101{\cdot}5}{D}(\beta_0 - \alpha_0)\right), \text{ where } \beta_0 - \alpha_0 = 139{\cdot}7''$$

$$= (1 - 0{\cdot}862) = 0{\cdot}138 \text{ ; and } m = 0{\cdot}069.$$

Example††. The following table shows the data relative to the elevations of the ground stations of a triangle PQR in a trigonometrical survey, the heights being respectively those of the instrument and the distant signal above the stations.

Side	Length (ft.)	Vert. angle	Height above station		Weight
			Theod.	Signal	
PQ	16030·2	$+16' 20''$	4·8	18·5	2
PR	14854·5	$-42' 24''$	4·8	12·4	2
QR	17245·0	$-49' 42''$	4·7	12·4	1
QP		$-10' 26''$	4·7	20·2	2
RP		$+48' 12''$	4·6	20·2	2

Determine the elevations of the stations, given that P is 362·42 ft. above datum. Adjust the elevations to close, allowing in particular for the fact that QR has not been observed reciprocally.

N.B. Whenever necessary, cosec 1″ may be taken as $2{\cdot}083 \times 10^5$, the coefficient of refraction 0·07, and $R \sin 1'' = 101{\cdot}5$ ft., R being the earth's mean radius in ft. (U.L.)

By axis-signal corrections from $\sigma_1 = \dfrac{s_2 - h_1}{D \sin 1''} \mp$ to $\pm$ angles :

$PQ, -178{\cdot}0''$; $PR, +106{\cdot}6''$; $QR, +93{\cdot}1''$; $QP, +201{\cdot}4''$; $RP, -218{\cdot}8''$.

For the stations observed reciprocally, curvature and refraction will be eliminated, while sufficient accuracy for the distances involved will be given by $H_2 - H_1 = D \tan \frac{1}{2}(\alpha_0 + \beta_0)$, where α_0 and β_0 are corrected for axis-signal.

Thus : $H_Q - H_P = 16030{\cdot}2 \tan \frac{1}{2}\{(16' 20'' - 2' 58'') + (10' 26'' + 3' 21{\cdot}4'')\}$
$= 63{\cdot}274$ ft.

Likewise $\quad H_P - H_R = 191{\cdot}705$ ft.

But $\quad H_Q - H_R = D \tan \left\{\alpha_0 - (1 - 2m)\dfrac{D}{2R}\right\}$,

where $\quad \alpha_0 = 49' 42'' + 1' 33{\cdot}1'' = 51' 15{\cdot}1'$

and $H_Q - H_R \quad = 251{\cdot}036$ ft.

If the elevation of P is assumed zero, the circuit gives :

Elevation $P = +63{\cdot}274 - 251{\cdot}036 + 191{\cdot}705$, with error $+3{\cdot}943$ ft.

Applying the method of correlates (p. 408),

$$\Sigma(e) = e_1 + e_2 + e_3 = -3{\cdot}943 \text{ ft.}$$

$$\lambda\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) = -3{\cdot}943 = \lambda(\tfrac{1}{2} + 1 + \tfrac{1}{2}) = 2\lambda : \quad \lambda - 1{\cdot}9715 ;$$

e_1 and $e_3 = \dfrac{\lambda}{w} = 0{\cdot}986$ and $e_2 = -1{\cdot}971$, giving the corrected differences :

$Q + R = +62{\cdot}288 + 190{\cdot}719 = +253{\cdot}007$, checking $-(251{\cdot}036 + 1{\cdot}971)$.

Whence the following elevations :

$$P = 326{\cdot}420 ; \quad Q = 388{\cdot}708 ; \quad R = 135{\cdot}701.$$

Example†. In a trigonometrical survey the computed horizontal distances from a Station P to Stations Q and R are respectively 3284 ft. and 31,160 ft., and the observed vertical angles from P are $-3° 16' 54''$ to Q and $+0° 48' 24''$ to R, Q and R being respectively 164·20 and 811·09 ft. above sea-level datum.

Determine the coefficient of refraction and the correct elevation of P, assuming the earth's mean radius to be 20,890,592 ft. ($\log R = 7{\cdot}3199507$). (U.L.)

Elevation Q from $P = 3284 \tan -(3° 16' 54'')$	$= -188{\cdot}30$
Curvature in PQ, $\dfrac{(3284)^2}{2R}$	$= +\ \ 0{\cdot}26$
	$-188{\cdot}04$
Given elevation of Q	$+164{\cdot}20$
Elevation of P, neglecting refraction	$+352{\cdot}24$
Elevation of R from $P = 31{,}160 \tan (0° 48' 24'')$	$= +438{\cdot}71$
Curvature in $PR = \dfrac{(31{,}160)^2}{2R}$	$= +\ \ 23{\cdot}24$
	$+461{\cdot}95$
Given elevation of R	$+811{\cdot}09$
Elevation of P neglecting refraction	$+349{\cdot}14$

The difference $352{\cdot}24 - 349{\cdot}14 = 3{\cdot}10$ ft. is due to refraction, and this may be assumed proportional to (distance)^2, and $\left(\dfrac{PQ}{PR}\right)^2 = 0{\cdot}0111$.

Letting $r_R - r_Q = 3{\cdot}10 =$ obs. diff. of refraction effects ; also

$$r_Q = 0{\cdot}0111 r_R \quad \text{and} \quad r_R(1 - 0{\cdot}0111) = 3{\cdot}10 ;$$

or

$$r_R = 3{\cdot}14 \quad \text{and} \quad r_Q = 0{\cdot}04,$$

$$\text{Coefficient of refraction} = \frac{\text{Refraction effect}}{2\,(\text{curve effect})} = \frac{3{\cdot}14}{2\,(23{\cdot}24)} = 0{\cdot}0675.$$

Elevation of P from R, corrected for refraction

$$= 349{\cdot}14 + 3{\cdot}14 = 352{\cdot}28.$$

Example††. The following notes refer to reciprocal trigonometrical observations between two triangulation stations A and B.

Stat.	Lat. N.	Long. E.	Elev. (ft.)	Heights (ft.)		Sight-ing	Mean obs. vert. angle
				Signal	Instr.		
A	44° 16′	74° 24′	411·98	29·0	24·2	B	+ 3° 42′ 12″
B	45° 28′	75° 36′		25·0	4·7	A	− 4° 55′ 34″

Assuming that the earth is a sphere of radius $R = 20{,}890{,}172$ ft. at sea-level datum, and that cosec $1'' = 2{\cdot}083 \times 10^5$,

(a) calculate the geodetic distance AB at mean sea-level ;

(b) determine *accurately* the elevation of B above M.S.L. datum. (U.L.)

(a) $\cos a = \cos b \,.\, \cos c + \sin b \,.\, \sin c \,.\, \cos A,$

that is, $\cos AB = \cos AP \,.\, \cos BP + \sin AP \sin BP \,.\, \cos APB,$

where co-lat. $AP = 45° \, 44'$, co-lat. $BP = 44° \, 32'$, and long. diff. $APB = 72'$.

log cos 45° 44′	$\bar{1}$·8438547 ;	log sin 45° 44′	$\bar{1}$·8549730
log cos 44° 32′	$\bar{1}$·8529936 ;	log sin 44° 32′	$\bar{1}$·8459188
log cos AP . cos BP	$\bar{1}$·6968483	log cos 72′	$\bar{1}$·9999047
cos AP . cos BP	0·4975632		$\bar{1}$·7007965
	0·5021072	sin AP . sin BP cos P	0·5021072
cos AB	0·9996704 ;	$\angle AB = 1° \, 28{\cdot}2667' = 1{\cdot}47111°$	

$$\text{Arc } AB = \frac{\angle AB \times R \times 2\pi}{360°}.$$

log $\angle AB$	0·1676452
log R	7·3199422
	7·4875874
log $360/2\pi$	1·7581226
log 536,370·4 ft.	5·7294648

$AB = 536{,}370{\cdot}4$ ft. $= 101{\cdot}5853$ miles.

(b) For notation, refer to Fig. 157, p. 363.

Correcting the angles for axis-signal, s_2 being the distant signal and h_1 the height of instrument :

$$\frac{(s_2 - h_1)}{D \sin 1''} \begin{matrix} - \text{ to } +\alpha \\ + \text{ to } -\beta \end{matrix} \quad \frac{(25{\cdot}0 - 24{\cdot}2)2{\cdot}083 \times 10^5}{0{\cdot}536370 \times 10^6} = -0{\cdot}31''.$$

$$\epsilon_2 = \frac{(s_1 - h_2)}{D \sin 1''} = \frac{(29{\cdot}0 - 4{\cdot}7)2{\cdot}083}{5{\cdot}36370} \quad +9{\cdot}46''.$$

Corrected angles : $\alpha_0 = +3° \, 42' \, 11{\cdot}69''$; $\beta_0 = -4° \, 55' \, 43{\cdot}46''$.

1st approximation :

$$H_2 - H_1 = D \tan \tfrac{1}{2}(\alpha_0 + \beta_0) = 536370{\cdot}4 \tan(4° \ 18' \ 57{\cdot}58'')$$
$$= 40{,}365{\cdot}57 \text{ ft.}$$

2nd approximation :

$$OA = R + H_1 \ ; \quad AOB = \theta = D/R \text{ rad.} \ ;$$

and $(OAB - OBA) = \{(90° + \alpha_0 - m\theta) - (90° - \beta_0 - m\theta)\} = (\alpha_0 + \beta_0)$. ......(1)

Now $$\tan \tfrac{1}{2}(\alpha_0 + \beta_0) = \frac{(R + H_2) - (R + H_1)}{(R + H_2) + (R + H_1)} \cot \tfrac{1}{2}\theta \ ;$$

or $$H_2 - H_1 = \tan \tfrac{1}{2}(\alpha_0 + \beta_0) \tan \tfrac{1}{2}\theta (2R + H_1 + H_2). \quad ......(2)$$

Since the distance AB is greater than 80 miles, it will not be sufficiently accurate to write the last term of (2) as $R\theta = D$, giving the 1st approximation (1) ; but this 1st approximation may be used in finding H_2, which can then be inserted in (2) for the required result, which by 7-figure logarithms is $H_2 - H_1 = 40{,}407{\cdot}93$ ft., giving the elevation of B, 40,819·91 above M.S.L. datum.

QUESTIONS ON ARTICLE 4

1†. The following notes refer to determining the mean elevation of a station P which has been interpolated by resection in a triangulation system, simple observations of equal weight being taken on three stations A, B, and C.

Station observed	Distance (ft.)	Vertical angle	Heights		Elevation
			Theod.	Signal	
A	16950	$+22' \ 20''$	4·8′	20·0′	772·60
B	15744	$+44' \ 35''$	4·8′	12·0′	874·00
C	14368	$-14' \ 05''$	4·8′	18·0′	604·50

Determine the mean elevation of Station P by correcting the observed angles, using a coefficient of refraction of 0·07 and assuming that one second of arc on the earth's surface = 100 ft. and that cosec $1'' = 2 \times 10^5$.

N.B.—Work to the nearest second and tenth of a foot; log tan $1'' =$ 6·6855749. (U.L.)

[Heights of P from A, B, and C respectively, calculated with log (α_0'' tan $1''$) : 671·1, 671·5, 671·9 ; mean, 671·5.]

2†. Two stations, A and B, are at a horizontal distance from one another of 37,529 ft. At A, a depression angle of 3° 4′ 2″ is recorded to B. Assuming that 101·3 ft. subtends 1″ at the earth's centre and that the correction for refraction is $\frac{1}{7}$ of that for curvature, calculate the difference in height of A and B, being given that the height of the theodolite was 4·7 ft. and that of the signal 12·7 ft.

Why should this measured depression angle from A to B be greater than the elevation angle from B to A? In what circumstances might it be possible for both angles to be measured as depressions? (U.L. Cart.)

[1990·0 ft.]

3. (*a*) Obtain an expression for the difference in level between any two points A and B, a considerable distance apart, by observations of reciprocal vertical angles. It is assumed that the distance between the two points is known. If this distance is not known, how may it be obtained?

Take into account the heights of instruments and targets. Illustrate your proofs with a large and clear diagram.

(*b*) Using the same diagram, obtain an expression from which the coefficient of refraction may be calculated.

(*c*) What is a usual value to assume for the coefficient of refraction if it has not been calculated? (U.B.)

4. (*a*) Obtain an expression for the difference of level between two points A and B, a considerable distance apart, B being the higher, by vertical angle readings from the point A. Take into account the height of the instrument at A and the height of the target at B. What is the assumption made in obtaining your equation for the difference of level?

(*b*) The mean vertical angle from A, reading on to a target at B, is 24′ 30″. The height of instrument at A is 5·00 feet and the height of the target at B is 4·00 feet. If the distance between the two points is 20 miles, assume the usual value for the coefficient of refraction and obtain the difference of ground level between the two points. The radius of the earth may be taken as 3,956 miles.

(*c*) Obtain an expression for the difference in level between two points by reciprocal vertical angle readings from the two stations. Heights of instruments and of targets should not be ignored. (U.B.)

[(*b*) 44,142 ft.]

5. (*a*) Find the difference in level between two points A and B by reciprocal vertical angle readings, given the following data:

Horizontal distance between the points A and B is 5,895·57 ft.

Vertical angle from A to B, $E_a = 01° 42' 02''$.

Vertical angle from B to A, $D_b = 01° 41' 46''$.

Height of instrument at $A = 4' 9\frac{1}{2}''$.

Height of instrument at $B = 4' 9\frac{1}{4}''$.

Height of target at $A = 5' 4\frac{3}{4}''$.

Height of target at $B = 5' 9\frac{1}{4}''$.

(*b*) Calculate also the height of B above A, using the vertical angle from A only and assuming 0·07 as the value of the coefficient of refraction (k).

Take $\log R = 7·321045$.

Prove the formula you employ in (*b*). (U.B.)

[(*a*) 174·60 ft. (*b*) 174·71 ft.]

6. At station A a signal, 15·3 feet high at B, reads $-0° 16' 35''$ on the vertical arc.

At station B a signal, 7 feet high at A, reads $+0° 22' 26''$. The ground height at A is 5,404 feet, the height of instrument at A is 5 feet and at B is 6·3 feet. AB is 46,778 feet. What is the ground level height at B? (T.C.C.E.)

[5128·94 ft.]

ARTICLE 5: PRECISE LEVELLING

The term **precise levelling** applies to a wide range of spirit levelling operations in connection with a State survey, river commission, city benchmark system, or an engineering project, primarily for the establishment of a system of primary and, possibly, secondary benchmarks.

In 1912, the International Geodetic Association, adopting the formulae of M. Lallemand, resolved that "levelling of high precision" should require that every line or set of lines, whether in circuit or not, should be run twice in opposite directions, on different days, so far as possible, and that the errors should not exceed specified limits when assessed by prescribed formulae; namely, ± 1 mm. $\sqrt{K}$ for probable accidental error and $\pm$ 0·2 mm. $\sqrt{K}$ for probable systematic error for lines not forming a net or for a net of not less than 10 polygons.

A probable error of ± 1 mm. per kilometre indicates a very high degree of precision, 2 mm. per km. a fair degree, and 3 mm. per km. a low degree, ± 5 mm. per km. being regarded unsatisfactory.

Instruments. Usually a reversible level is employed in order that instrumental errors may be eliminated by systematic reversals of the telescope. In the earliest work, the Y-level was used, and later, particularly in America, the Kern pattern, while the Zeiss No. III model was used throughout the Second Geodetic Levelling of England and Wales (1912–20). Recently the Ordnance Survey has adopted the Geodetic Level of Messrs. Cooke, Troughton & Simms for primary levelling operations. This instrument embodies the reversible feature formerly inherent in the improved reversible level. At present a very popular model is the pattern designed by the U.S. Coast & Geodetic Survey. The pattern as made by Messrs. Cooke, Troughton & Simms has been used on various surveys (Fig. 158).

This introduces many interesting features; in particular, provision for changes of temperature both in the bubble and telescope, a cloth cover being fixed over the telescope tube. The micrometer tilting screw and

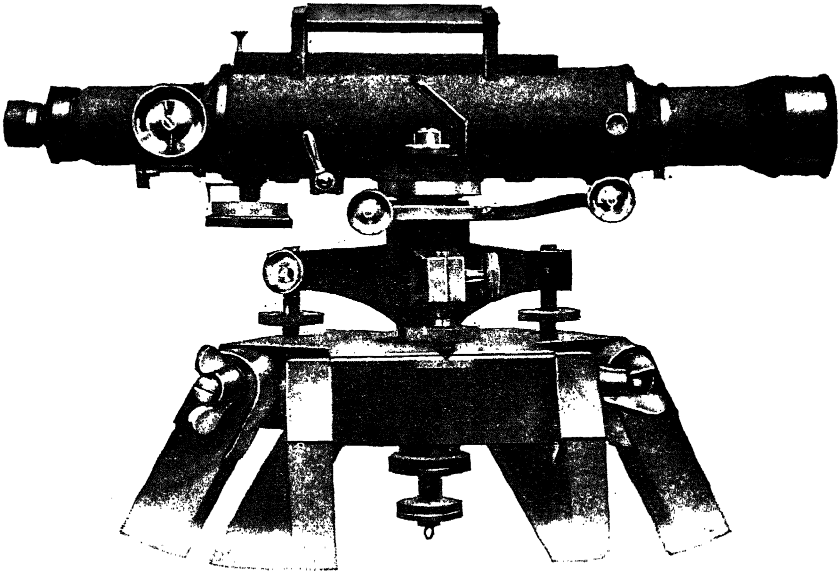


FIG. 158.

relief are included, also a circular bubble for preliminary levelling. Messrs. E. R. Watts & Co. also construct the model with their well-known patent constant bubble. Sometimes the parallel plate micrometer is used (p. 13), also stadia lines are provided.

Although the use of well-seasoned wooden staves is giving way to patterns with invar insets, the effects of expansion were considered from the earliest days of precise levelling, as the cumulative temperature effect becomes very considerable in hilly country when a number of short back-sights and foresights are necessarily involved. Also it is a part of the routine to check the graduations against a standard bar from time to time, while an important source of error is wear or displacement of the shoe of the staff. Commonly pairs of staves are used, these being provided with bubbles, plummets, handles, and steadying poles.

In recent years the 10 ft. (3 m.) staff with an invar steel strip, $\frac{1}{2}$ in. wide inset in the front of the staff, has been used, the strip being fixed only at the shoe, so as to be independent of alteration in length of the wooden body. The smallest division is $\frac{1}{50}$ ft. for use with the parallel plate micrometer, as in the case of the Cambridge staff, as used by the O.S. (1912–20). Foot plates are used as turning points ; sometimes a steel pin with a spherical head, the shank being driven into the ground rigidly up to a base plate.

Benchmarks. The construction of these will depend largely upon the extent and allowable cost of the survey. Wherever possible, they should be in direct contact with a rigid stratum or connected therewith by means of a concrete pillar cast *in situ.*

Procedure. In recent years there has been a desirable tendency to introduce the methods of precise levelling to engineering projects; and the aim of this article is to deal with these and the work of extensive levelling operations. Hence the following general instructions are advanced with the suggestion that the surveyor should consult the report of some noteworthy scheme; as, for example, the General Instructions for Precise Levelling, U.S. Coast and Geodetic Survey, from which much of the following matter has been drawn.

(1) The work shall be carried out during . . . (specified hours) when the atmospheric conditions are favourable and the refraction effects less variable. The instrument shall be sheltered by a screen or umbrella from the direct effects of the sun and the wind, protection from the sun being desirable when the instrument is in use at stations and in transport from station to station. Observations shall not be taken in high winds or in rainstorms.

(2) All lines shall be levelled independently in both directions, and change points and level stations shall not be common to each series. Backward lines should be run under different atmospheric conditions from those prevailing for the forward lines.

(3) The sight length shall not exceed a specified distance (say 300 to 450 ft.) depending upon the instrument, and the maximum permissible length shall only be used in the most favourable circumstances.

(4) Back- and foresight distances should be equal, or corrections should be applied for curvature and refraction. (In lower-grade work, back- and foresight distances balanced in the aggregate will be accepted). When the stadia lines are used to facilitate the balancing of sight lengths, the mean of the three horizontal hair readings should be taken instead of the single central hair reading. (In no case should the difference between the length of a backsight and a foresight exceed . . . ft.)

(5) Once during each day in the field the collimation adjustment should be verified. Otherwise all readings shall be taken with the telescope in the following positions . . . (normal and inverted in the case of Zeiss pattern levels, adjustment having been made for the mean position).

(6) Two similar staves shall be used alternately for backsights and foresights, and the staves shall be examined and compared with a standard bar at periods of . . . . A staff bubble and plummet shall be used.

(7) All readings shall be taken on the . . . (specified) pattern staff, the parallel plate micrometer being used if prescribed. The staff shall be held on the . . . (specified) form of footplate, or turning point, at all staff stations, and the plates shall not remain for lines in the opposite direction.

At alternate stations of the level, the backsight shall be taken before the foresight, and *vice versa*, in order to reduce the effects of settlement of the tripod. (The staff thermometer shall be read at all stations in the case of wooden staves.)

(8) If in any section between benchmarks the error exceeds $c\sqrt{M}$ and $c'\sqrt{K}$ where c and c' are coefficients and M and K are distances between the benches in miles and kilometres, both the backward and forward lines shall be run again.

. Write a brief note on the discrepancies between the results of the First and Second Geodetic Levels of England and Wales. Describe concisely the benchmarks of the latter operations.

Ordnance benchmarks. In the First Geodetic Levelling of England and Wales (1840–1860), it was intended that a mark on Mersey Docks, Liverpool, should coincide with mean sea-level. Since this was of disputed accuracy, the work falling short of later conceptions of precision, the Second Geodetic Levelling (1912–20) was carried out in conjunction with tidal observations at Dunbar, Newlyn, and Felixstowe. Briefly, it was concluded that any differences are small, the old datum being practically the same as M.S.L. around the coast and 0·13 ft. above M.S.L. at Newlyn : the new levels will not differ greatly from the old ones at Liverpool, but will show a maximum difference of about 1·75 ft. in the eastern counties, the later values being less than the original. The Zeiss No. III level in conjunction with a Cambridge staff with invar strip was used throughout the work.

The benchmarks were of three classes :

1st class B.M.'s are all on solid rock, at an average distance of 25 miles apart, the precise levelling between these forming a basis for (*a*) branch levelling, and for (*b*) determining relative movement, if any, between the levels of the land and sea. The overall dimensions are about 5′ 6″ × 3′ 0″. Outside, in the tapered portion, a gun-metal bolt is inserted in a granite pillar, 12″ above the ground ; and, buried under a cover stone, and further protected by iron covers at a total depth of 3′ 0″, are comparison points in granolithic concrete ; one a gun-metal bolt and the other polished flint.

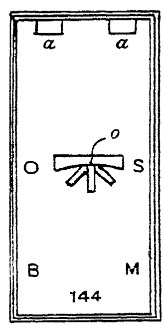


FIG. 159.

2nd class B.M.'s are flush plates, fixed about one mile apart on old buildings, bridge piers, etc., each plate being about $7\frac{1}{2}'' \times 3\frac{3}{4}''$, with a serial number, as shown in Fig. 159. A detachable staff bracket is used for supporting the staff. Hook ends of this bracket are attached at *aa*, while the horizontal bracket rests on the reference point *o*, the levelling thumbscrew

bearing near the number until the attached spirit bubble becomes central.

3rd class B.M.'s are copper rivets let into a horizontal surface of stone or brick, at distances averaging $\frac{1}{4}$ mile apart.

QUESTIONS ON ARTICLE 5

1††. Prepare a general scheme of instructions for precise levelling in such a form that it can be modified to meet the requirements of any given benchmark system.

2†. Discuss the objects and practice of precise levelling, describing briefly the instruments employed.

A circuit of benchmarks, approximately 6 miles in perimeter, was run under the same conditions three times with respective errors of closure of 0·032, 0·040, and 0·036 ft., the level having been set up 49 times in each case with balanced back- and foresights.

Assuming that the error of sighting the staff is constant for sight lengths between 250 and 350 ft., give a rule for running circuits under similar conditions, expressing the linear distance in miles. (U.L.)

[Here the average error is 0·036 ft. (or 0·0362 m. s. e.) in 49 settings up with an average sight length of 323 ft. in exactly 6 miles.

[$0{\cdot}036 = c\sqrt{6}$; or $E = 0{\cdot}0147$ ft. $\sqrt{M}$

GEODETIC METHODS

The figure of the earth. Although in geodetical surveying the figure of the earth is regarded as spherical, in order to admit the direct application of spherical trigonometry, its precise form more closely approximates to that of an oblate spheroid, the surface of which is determined by the mean surface of the sea, in so far as an exact surface of revolution could exist in the fact of the irregular distribution of mass.

Strictly, however, the figure of the earth is termed the **geoid**, which has the characteristic that its surface at any point lies in a plane tangential to the direction of gravity at that point. Such a figure is by no means regular, since the direction assumed by a plumb line is dependent upon numerous factors, particularly the proximity to oceanic and mountainous masses.

The oblate spheroid is the regular geometrical solid most closely approximating to the geoid ; and the surface of the spheroid is generated by the rotation of an ellipse about its minor axis, which in the present connection is styled the **polar axis**, being the axis about which the earth

makes its diurnal rotation. Since the ratio of the major and minor axes of this ellipse of revolution is very nearly unity, being about 301 to 300, the application of the spherical approximation is obvious. There are, however, certain geodesical and geophysical operations in which the approximation is insufficiently accurate and the geometry of the spheroid must be introduced.

Latitude. Were the earth a perfect sphere, the latitude of a point on its surface could be generally defined by what is termed **geocentric latitude**, which is the angle between that point, the earth's centre, and the plane of the equator, as $MOE = \lambda = \phi$, in Fig. 160, where WE is the trace of the equatorial plane. But the so-called **geographical latitude**, as understood in astronomy and cartography, is the angle between the prolongation of a plumb line suspended over the point and the plane of the equator, as indicated also in Fig. 160 by $A'O'W' = \lambda'$, $A'O'$ being the direction determined by the plumb line, or mathematically, by the normal to the ellipse at A'.

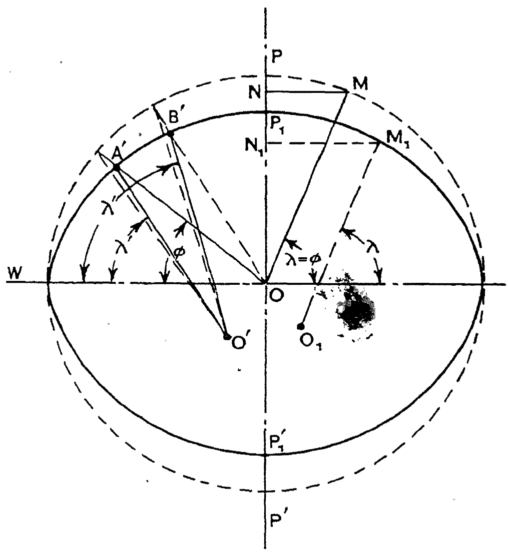


Fig. 160.

Now the difference between the values of the geocentric and geographical latitudes, ϕ and λ, of a point affords a means of determining the true form of the earth's surface. For if observations for latitude were made at the points A' and B', the difference would be known, while D, the geodetic distance $A'B'$, might be determined by surveying; then if O'

be taken as the centre of curvature for the middle point of $A'B'$, $O'A'$ or $O'B'$ would be the radius of curvature ρ of the meridian for the short length $A'B'$ very nearly ; that is,

$$\rho = \frac{D \times 360°}{2\pi(\lambda'' - \lambda')}.$$

Observations such as these would show that the radius of curvature $ds/d\lambda$ increases towards the poles and decreases towards the equator, confirming the assumption of a spheroidal surface.

Longitude. The longitude of a point may be defined as the arc of the equator between the meridian of Greenwich and the meridian of observation. Thus if the earth were a perfect sphere, the length of a degree on a parallel of latitude might be computed by multiplying the corresponding length of 69·17 statute miles by the cosine of the latitude ; that is, 69·17 cos λ miles. But the value of a degree on a given parallel differs for the spherical and spheroidal surfaces, as indicated in Fig. 160, where a circle of radius M_1N_1 is the parallel for a latitude λ through a point M_1 on the spheroid, and a circle of radius MN the parallel for the same latitude M on the sphere. The inherent discrepancy suggests the means of determining the actual length of a degree of longitude in various latitudes and comparing this with the corresponding lengths as computed by 69·17 cos λ miles.

Mathematical analysis. The prime dimensions of the spheroid are derived from the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \dots\dots\dots\dots\dots\dots\dots\dots(1)$$

where a is the semi-major axis, or equatorial radius of the meridional ellipse and b the semi-minor axis or polar radius, x and y being the coordinates of a point such as A' or M_1 with respect to these axes and an origin at the earth's centre.

In geodesy, it is customary to introduce the relation between the axes and the ellipticity (or compression) $e = \frac{a - b}{a}$ and the eccentricity $\sqrt{a^2 - \cdots}$

Now the radius of curvature ρ along the meridian at any point,

$$\rho = \left\{ \frac{1 + (dy/dx)^2}{\cdots} \right\}^{\frac{3}{2}} = \left\{ \frac{a^4 - x^2(a^2 - b^2)}{a^4 b} \right\}^{\frac{3}{2}}. \quad \dots\dots$$

Also since M_1O_1 is perpendicular to the tangent at M_1,

$$\tan\lambda = -\frac{dx}{dy} = \frac{a^2 y}{b^2 x} \text{ (from (1)) and } y = \frac{b^2 x \tan\lambda}{a^2}; \quad \dots\dots\dots(3)$$

and, on substituting for y from (1),

$$\frac{a^4}{\text{an}^{\circ\,\circ}}, \quad \text{where} \quad x = M_1 N_1 \qquad .(4)$$

Hence it follows from (2) that

$$\rho \quad \frac{}{(b^2 \tan^2\lambda + a^2)^{\frac{3}{2}}} - - \frac{a^2(1-\epsilon^2)}{(1-\epsilon^2 \sin^2\lambda)^{\frac{3}{2}}}, \quad \ldots\ldots\ldots\ldots \qquad .(5)$$

which gives the radius of curvature in terms of the semi-axes and the geographical latitude.

Incidentally, x in (4) is the radius on any parallel of latitude, whereas for the sphere, $a=b$ and $x=a \cos \lambda$; and thus it is possible to compare computed values of a degree of meridian with observed values.

If the geocentric latitude ϕ is introduced, $\tan \phi = y/x$, while (from (4)),

$$\tan \lambda = \sqrt{\frac{a^2 - x^2}{x^2(1-\epsilon^2)}},$$

leading to the constant relationship between λ and

$$\tan \phi = \frac{\upsilon}{a^2} \tan \lambda = (1-\epsilon^2) \tan \lambda. \qquad .(6)$$

Also the radius of curvature perpendicular to the meridian,

$$\eta = \frac{a}{\sqrt{1-\epsilon^2 \sin^2\lambda}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

as would follow from producing the normal to the ellipse at A' or M_1 to meet the minor axis, the point thus determined being that through which the normal always passes as the ellipse rotates.

Finally, if ω is the angle which a plane containing the polar axis makes with the meridian, the radius of curvature at the section determined by the plane is

$$\frac{\rho\eta}{\rho \sin^2\omega + \eta \cos^2\omega}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Dimensions of the earth. A historical note with a summary of the values of a, b, and e would be out of place in a work of this nature. Sufficient therefore to say that the chief authorities in these investigations were Bessel, Everest, and Clarke, although considerable research has been made in recent years, particularly in regard to modifying the values of these investigators.

The Central Bureau of the International Geodetic Association adopted the value $a = 6{,}377{,}397{\cdot}155(1+0{\cdot}0001)$ metres, with $e = 1/299{\cdot}15$, following Bessel's measurements of 1841. These lead to values of $a = 20{,}925{,}221$ ft.

and $b = 20{,}855{,}272$ ft., the international metre being equal to 3·2808257 ft. The U.S. Coast and Geodetic Survey adopted Clarke's spheroid as its standard in 1881, but in 1906 decided that the best values for the United States would be $a = (6{,}378{,}283 \pm 74)$ metres with $e = \dfrac{1}{297 \cdot 8 \pm 0 \cdot 9}$.

Everest's first constants were adopted for the India Survey: $a = 20{,}922{,}932$ ft. with $e = 1/300 \cdot 8$, the second constants not being utilised.

In examples, particularly those occurring in examination questions, it is impossible to adhere to rigid values of a and b for the spheroid and R for the sphere, and in consequence discrepancies will frequently occur.

The following formulae for Clarke's spheroid are useful in deducing the length corresponding to a minute of latitude m on the meridian and n on a great circle perpendicular to the meridian.

$$m = 6076 \cdot 76 \left(1 \pm \frac{\sin 2\delta}{200}\right) \text{ ft.}; \quad n = 6076 \cdot 76 \left(1 + \frac{1}{300} \pm \frac{\sin 2\delta}{600}\right) \text{ ft.},$$

where δ is the difference between the latitude λ and 45°, latitudes greater than 45° taking the plus sign. Accordingly the length corresponding to one minute of longitude in latitude λ may be taken as

$$6087 \cdot 15 \sqrt{\frac{1}{1 + 0 \cdot 006724 \tan^2 \lambda}}.$$

The radius R of the sphere is sometimes taken (a) as the mean value for the area covered by the survey, (b) as the mean of the polar and equatorial radii (20,888,629 ft., or 20,890,543 ft.), or (c) as that of a sphere with a surface area equal to that of the earth (20,901,581 ft.). When R is taken at 20,890,500 ft., or 7913 statute miles, the length of a minute of arc is 6077 ft., while if R is assumed to be 20,901,581 ft., the corresponding value is 6080·27 ft., or 1·15157 statute miles.

ARTICLE 6: MERIDIANS AND PARALLELS

The following problems may be considered appropriately in the present article: (I) convergence of meridians; (II) parallels of latitude; (III) latitude and longitude by account. It is, therefore, essential to state a rigid distinction between azimuths and bearings.

Azimuth and bearing. The **azimuth** of a line may be defined as the angle between two great circles of the terrestrial sphere, one of these circles being the meridian through a point on the line and the other containing the line itself.

Now in the case of a line AB, the meridians through A and B are parallel only when A and B are on the equator, when the azimuth of B from A is equal or supplementary to the azimuth of A from B. In general, however, the meridians are not parallel, but converge to the poles, and, in consequence, the azimuth from B to A is neither equal nor supplementary to the azimuth of B from A. (See Note i, below.)

The **bearing** of a line is measured from a plane of reference at each station, and this is parallel to some standard plane, preferably near the centre of the area under survey; and, in consequence, the forward and backward bearings of the line are supplementary angles.

Thus a straight line run with the theodolite is actually an arc of a great circle with bearings that change at every point in its length, while a line run to a constant bearing or angle would be a curved line on the terrestrial sphere, a line constantly at 90° to the meridians being a parallel of latitude. The straight line, or great circle course, is the shortest distance between points on the sphere. (See Note ii.)

Now if a line AB were set out in a mean latitude λ in a direction approximately east and west, the azimuth of the first course might be used as a bearing from which the bearings of the succeeding lines are reduced from observed angles; and if the bearing of the last line is computed in this manner, it will be found not to agree with the azimuth of this line as determined by independent observation. The discrepancy is due to convergence. (See Note ii.)

Note i. **Convergence** is in no way related to the distortion concomitant with neglecting the spherical form. It is an actual fact which would become evident if, say in latitude 45°, an angle of 90° were set out from the meridian at A, and 60 nautical miles were run in this direction to B, for then the line AB would be found to make an angle of 89° with the meridian determined at B.

Note ii. Convergence is evident in most map projections, though not in Mercator's and other cylindrical nets. It may be seen on the 6-in. Ordnance maps, the degrees, minutes, etc., of longitude along the upper margin occupying smaller distances than the corresponding divisions on the lower margin. It is thus necessary to find equal longitudes on the upper and lower margins in order to insert the true meridian.

Note iii. A line of constant bearing is represented in Mercator's projection as a straight line, called a **rhumb line** or **loxodrome**, and this facilitates the navigation of a ship or aeroplane between two points, although the actual running to a constant bearing is along a curved course. **Gnomonic projections** also possess a similar property.

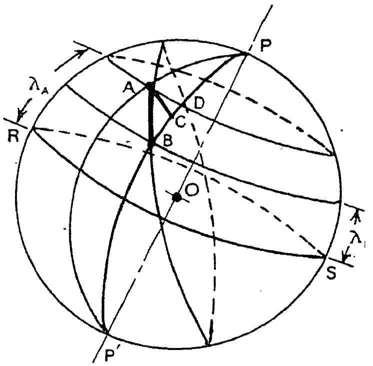


FIG. 161.

(I) **Convergence of meridians.** Convergence may be defined as the effect of employing reference meridians as parallel lines over a considerable extent of the earth's surface.

Suppose a traverse or triangulation survey commences with a line Ab, and the azimuth or bearing of this line from the meridian be determined, then after working through a considerable distance, with the meridian of A effectively a standard meridian, it becomes desirable to check the accuracy of the work by means of an observation of the azimuth of a line, qB, say. Now the bearing of qB with reference to an axis through B parallel to the original meridian at A is calculated by co-ordinates, whereas the azimuth of qB is not referred to that axis, but to a meridian through B, which, converging, meets the original meridian at the poles. Hence in order to check the work, it is necessary to calculate the " change in azimuth ", or angle between the true meridian through B and the line through B parallel to the original meridian at A.

Let the latitude of $A = \lambda_A$, of $B = \lambda_B$ and let the longitude difference between A and B be θ in angular units.

Then the convergence k is the difference between 180° and $(A+B)$; that is, $k = 180° - (A+B)$, where A and B are angles of the spherical triangle ABP. (Fig. 161.)

Now $P = \theta$; $AP = 90° - \lambda_A$, and $BP = 90° - \lambda_B$.

Then by the usual formula,

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}\theta.$$

$$\cot \{90° - \tfrac{1}{2}(A+B)\} = \frac{\cos \frac{1}{2}(\lambda_A - \lambda_B)}{\sin \frac{1}{2}(\lambda_A + \lambda_B)} \cot \tfrac{1}{2}\theta.$$

Whence
$$\tan \tfrac{1}{2}k = \tan \tfrac{1}{2}\theta \, \frac{\sin \frac{1}{2}(\lambda_A + \lambda_B)}{\cos \frac{1}{2}(\lambda_A - \lambda_B)} . \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

When AB is small compared with the radius of the earth, the tangent of the angle k is very nearly equal to the angle in circular measure, or

$$k = \frac{\sin \frac{1}{2}(\lambda_A + \lambda_B)}{\cos \frac{1}{2}(\lambda_A - \lambda_B)} \theta \text{ radians}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

which is likewise true when both k and θ are expressed in minutes and seconds, since $k' \tan 1' = k$ radians, and $\theta \tan 1' = \theta$ radians.

Also if R is the radius of the earth, l the linear difference of latitude, and d the linear difference of departure, then in circular measure,

$$\lambda_A - \lambda_B = l/R, \ldots\ldots\text{(i)} \quad \text{and} \quad \theta = \frac{d}{R\cos\frac{1}{2}(\lambda_A + \lambda_B)}. \ \ldots\ldots\text{(ii)}$$

Whence (2) becomes

$$k = \frac{d \,.\, \tan\frac{1}{2}(\lambda_A + \lambda_B)}{R \,.\, \cos\frac{1}{2}(\lambda_A - \lambda_B)} \text{ radians}, \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(3)}$$

division on the right by $\tan 1'$ or $\sin 1'$ giving the convergence in minutes.

When the difference of latitude is relatively small, $\cos\frac{1}{2}(\lambda_A - \lambda_B) = 1$, and it follows that

$$k = \theta \sin\frac{1}{2}(\lambda_A + \lambda_B) = \Delta \tan\frac{1}{2}(\lambda_A + \lambda_B),$$

which are the well-known approximate rules for convergence in minutes :

$$k' = \theta' \sin(\text{middle lat.}) = \Delta' \tan(\text{middle lat.}), \ \ldots\ldots\ldots\text{(4), (5)}$$

Δ' being the angular value of the departure, $\dfrac{d}{R\tan 1'}$ minutes.

Note. Although Eq. 2 is the form usually given in the present connection, many prefer to use the cosine and sine rules, as, for example : Calculate the side AB from the given sides AP and BP and the included angle P, and then determine the angle B thus :

$$\cos AB = \cos AP . \cos BP + \sin AP . \sin BP . \cos P$$

$$\sin B = \frac{\sin AP}{\sin AB} \sin P. \quad \text{(Fig. 161).}$$

Sphere and spheroid. In the foregoing discussion, the earth has been regarded as a sphere, with m and p the respective lengths of $1'$ of meridian and parallel ; 6082 and 6082 $\cos\lambda$ ft. conveniently. Hence the angular changes in latitude, departure, and longitude are respectively,

$$\delta\lambda = l/m \ ; \quad \Delta' = d/m \ ; \quad \text{and} \quad \theta' = \frac{d}{m\cos\lambda}.$$

Frequently problems arise in which the spheroidal dimensions are introduced, the mean tabular values of m and p being involved. These values are given for $5'$ differences of latitude, between which interpolation is used. Also in this connection, the symbols for small elements are introduced, $\delta\theta$ being a small change in longitude and $\delta A = k$, the corresponding change in azimuth. Although, strictly, spherical trigonometry no longer applies, the convergence rule, $\delta A = \delta\theta$ (sin middle lat.), is used in conjunction with

$$\delta\lambda = \frac{L}{m}\cos(A + \tfrac{1}{2}\delta A) \quad \text{and} \quad \delta\theta = \frac{L}{p}\sin(A + \tfrac{1}{2}\delta A),$$

L being the geodetic distance between the stations.

Convergency formulae are used in two connections, and then over a limited extent of surface : (1) Correcting bearings in route surveying, and (2) correcting the parallels in the U.S.A. Land Surveys. Otherwise the general problem of geodetic distances is involved. Control points might be established by observations of latitude λ_A and λ_B at the end stations A, B, in conjunction with azimuth observations at those stations. Also the longitude difference might be determined by an observation for time, giving local mean time as compared with a chronometer set to G.M.T. With these data the spherical triangle can be solved, if the spherical figure is within the limit of error.

When, however, great distances are involved, the problem is exceedingly complex, introducing the method of Puissant, which should also be used when great precision is required. For comparatively short distances, not exceeding about ten miles, the approximate method of middle latitudes (as suggested in (4), p. 381) may be used, the rule also applying to the spheroid when the total distance is very small in comparison with the earth's radius. Frequently successive approximations are used in applying the method to the spheroid.

(II) **Parallels of latitude.** When great distances concomitant with a high degree of accuracy are concerned, the setting out of a parallel is a complex problem, although theoretically a parallel would follow from a series of short lines, each having an azimuth of 90°.

Two cases will be considered in this article : (*a*) short lengths on the sphere or spheroid, and (*b*) great lengths on the sphere.

(*a*) The method applicable to this case is used in the **Township Areas** of the rectangular system of Land Surveys of the U.S.A. The areas within the standard parallels and guide meridians are divided into townships of as nearly six miles square as the form of the earth will permit, and these are subdivided by meridians and parallels into 36 sections each of approximately 640 acres. Within these townships, convergency tables are used, giving corrections to the parallels, change in azimuth, difference in latitude, expresssed in arc, etc., for a range of latitude from 30° to 70° N.

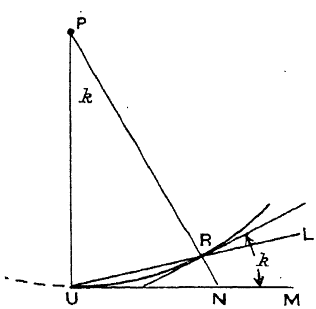


FIG. 162.

These are based upon the rules, $k = \theta \sin \lambda$ or $\dfrac{L}{m} \tan \lambda$, L being the linear distance.

The process, which is shown in Fig. 162, may be applied to single distances not exceeding 10 statute miles. Here k is the increase in azimuth from a great circle perpendicular to the meridian at U, UL being a great circle through points U and R on the parallel.

Thus $LUM = \frac{1}{2}k$, and offset $RN = \frac{1}{2} \, . \, UN \, . \, k \tan 1''$.

But $$k = \frac{L \tan \lambda}{R \tan 1''}, \quad \text{and} \quad UN = L\,;$$

hence $$RN = \frac{L^2 \tan \lambda}{2R}, \quad \text{or} \quad \frac{L^2 \sin \lambda \tan 1''}{2p},$$

where p is the length of $1''$ of parallel in the latitude λ.

Strictly the offset RN is along the meridian RP, and if a meridian is determined at R, the process may be repeated for a further distance, setting out the line at 90° to the meridian RP.

(*b*) When the earth's figure is assumed to be spherical, the meridians at the beginning and end of the parallel AD in Fig. 161 form an isosceles triangle with a great circle through A and D, the arc passing very close to the required parallel. If now a meridian were set out through the middle point of the arc of the great circle, two right-angled triangles would be formed, and in this way the latitudes of points on the arc AD of the great circle might be determined. From the differences of these latitudes and the constant latitude, the offset distances to points on the actual parallel can be computed. Usually the length of the parallel L or the corresponding difference of longitude would be given, as in the example of p. 385.

(III) **Latitude and longitude by account.** The latitude and longitude of a station may be accounted by adding algebraically the latitudes and departures of the intervening courses to the observed (or deduced) latitude and longitude of a previous station of the survey.

When the differences of latitude and longitude in a route survey are small, not exceeding five miles, a first approximation may be made by taking $1'$ of arc equal to 1·151 statute miles of latitude and 1·153 statute miles of longitude on the equator. Since, however, the length of $1'$ of longitude varies from 0 at the pole to 1·15287 statute miles on the equator, it is necessary to reduce the departures to equivalent distances on the equator before reducing to arc; that is,

Long. diff. in minutes, $$\theta' = \frac{d}{R \cos \lambda \, . \tan 1'}$$

and this reduces respectively to

$$\theta' = d \, . \sec \lambda, \quad \text{and} \quad \theta' = \frac{d \sec \lambda}{1{\cdot}15287},$$

according as nautical or statute miles are considered.

Thus, if the departure on the 45° parallel is 5 statute miles, the difference on the equator is 5 sec 45° = 7·0710 miles, and this corresponds to an angle θ of $6'\,8''$.

When either a second approximation is required or the latitude difference exceeds 5 miles, the following rule for Clarke's spheroid may be used :

$$m = 6076{\cdot}76\left(1 \pm \frac{\sin 2\delta}{200}\right),$$

where m is the length of $1'$ of arc in feet and δ the difference between latitude λ and $45°$, latitudes greater than $45°$ taking the *plus* sign.

Example†. In a preliminary survey the notes are reduced to the following traverse :

AB, 8 miles at N. 76° E. ; BC, 6 miles at N. 71° E. ; and CD, 9 miles at N. 65° E., the bearing of AB being deduced from an azimuth observation and those of BC and CD from deflection angles. State what correction must be applied to the reduced bearing of CD at D to allow for the convergence of meridians.

The latitude of A is 56° N. and the mean radius of the earth 3,916 miles. (U.L.)

Latitude diff. l between A and D,

$$l = 8 \cos 76° + 6 \cos 71° + 9 \cos 65° = 7{\cdot}6923491 \text{ miles.}$$

$$\lambda_D - \lambda_A = \frac{1}{R \tan 1'} \text{ min.} = \frac{7{\cdot}6923491}{3916 \times 0{\cdot}0002909} = 6{\cdot}733$$

Thus $\lambda_D = 56°\ 6{\cdot}733'$.

Departure diff. between A and D,

$$d = 8 \sin 76° + 6 \sin 71° + 9 \sin 65°$$

$$= 21{\cdot}5934474 \text{ miles.}$$

Convergence
$$k = \frac{d \,.\, \tan \frac{1}{2}(\lambda_A + \lambda_D)}{R \tan 1' \cos \frac{1}{2}(\lambda_A - \lambda_D)} \text{ min.}$$

$$= \frac{21{\cdot}593447 \tan \frac{1}{2}(56°\ 3{\cdot}334')}{3916 \times 0{\cdot}0002909 \cos(3{\cdot}334')}$$

$$= 28{\cdot}16'.$$

A boundary between two provinces is to be a line 40 statute miles in length along the 55° 30′ parallel of latitude.

Submit the calculations relative to setting out the boundary monuments at 10, 20, and 30 miles by meridional offsets, assuming the mean radius of the earth to be 20,890,172 feet. [log $R = 7{\cdot}3199422$.] (U.L.)

(In this, as in many examples, the logarithmic calculations are omitted.)

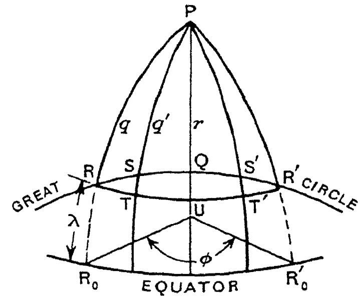


FIG. 163.

Length of parallel,

$RR' = R\phi \cos \lambda$

, RR' .

$$\frac{211200}{20,890,172 \cos 55° 30' . \tan 1'}$$

$= 61{\cdot}36173'$.

Now ϕ is equal to the angle at P (Fig. 163), and the problem is expediently reduced to two parts of an isosceles spherical triangle, which by symmetry are two similar right-angled triangles RPQ, $R'PQ$, in which $P = \frac{1}{2}\phi = 30{\cdot}68'$.

The following sides and angles are required in the solution, which may be facilitated by means of Napier's circle (p. 263).

(*a*) R, $p = RQ$, and ?. (*b*) S, $p' = SQ$, and $q' =$?

$$\cos q = \frac{\sin 55° 30'}{\tan 30{\cdot}68'},$$

giving $R = 89° 34{\cdot}7146'$, which is the angle from the meridian at R.

° 30′. $\sin 30{\cdot}68'$, giving $p =$]

$$\cos r = \frac{\cos q}{\cos p} = \frac{\cos 34° 30'}{\cos 17{\cdot}3819'}, \text{ giving } r = 34° 29{\cdot}9355'.$$

In the triangle PQS,

$$\cos S = \sin \tfrac{1}{2}P . \cos r = \sin 15{\cdot}34' . \cos 34° 29{\cdot}9355',$$

giving $S = 89° 49{\cdot}3545'$.

$$\cos p' = \frac{\cos \frac{1}{2}P}{\sin S} = \frac{\cos 15{\cdot}34'}{\sin 89° 47{\cdot}3545'}, \text{ giving } p' = 8{\cdot}6667'.$$

$\cos q' = \cos r . \cos p' = \cos 34° 29{\cdot}9355' . \cos 8{\cdot}6667'$, giving

$$q' = 34° 29{\cdot}9516'.$$

Offsets at Q and S in angular units are $0{\cdot}0645'$ and $0{\cdot}0484' = \omega$; and

$$QU = \frac{\pi R \omega}{180 \times 60} =$$

$$\log R = 7{\cdot}3199422$$

$$\log \frac{180 \times 60}{} = 3{\cdot}5362739$$

$$3{\cdot}7836683 = \log k.$$

Adding log 0·0645 and log 0·0484 to log k, the following values of the offsets are obtained :

$$QU = 391{\cdot}9475 \text{ ft.}, \quad ST = 294{\cdot}1126 \text{ ft.}$$

Similarly, the distances $RQ = p$ and $SQ = p'$,

$$p = 105624{\cdot}727 \text{ ft.}, \quad p' = 52665{\cdot}000 \text{ ft.}$$

Example††. In a network of major triangulation, a station A is in lat. 44° 52′ 12″ N. and long. 42° 24′ 15″ E. and an adjacent station B is in lat. 45° 10′ 42″ N. and long. 43° 8′ 45″ E.

As a first approximation, the earth was assumed to be a sphere with a radius R of 20,890,172 ft., and a re-calculation was made with the following values for the spheroid :

Lat.	1″ lat. (m ft.)	1″ long. (p ft.)
45° 00′	101·2804	71·8607
45° 05′	101·2819	71·7566

Calculate the length of the side AB, (a) for the sphere, and (b) for the spheroid. [log $R = 7{\cdot}3199422$.] (U.L.)

(a) Assuming a spherical triangle APB, P being the north pole,

$$\cos AB = \cos AP . \cos BP + \sin AP . \sin BP . \cos P,$$

where colat. $AP = 45° \, 7' \, 48''$, colat. $BP = 44° \, 49' \, 18''$, and long. diff.

Whence $\cos AB = 0{\cdot}9999435$, and $\angle AB = 0{\cdot}609375° = 36{\cdot}5625'$.

$$\text{Arc } AB = \frac{\angle AB \times R \times 2\pi}{360°} = 222{,}169{\cdot}65 \text{ ft.} = 42{\cdot}0805 \text{ miles.}$$

(b) Here the station A may be regarded as the origin of rectangular co-ordinates with $B = m \, \delta\lambda$ and $p \, \delta\theta$ ft. respectively N. and E. of A, where m and p are the values of 1″ of lat. and long. respectively, the differences of which are $\delta\lambda$ and $\delta\theta$ accordingly (Fig. 164).

Thus $\delta\lambda = 18' \, 30''$ and $\delta\theta = 44' \, 30''$, with a middle lat. of $\lambda = 45° \, 1' \, 27''$.

Also $m = 101{\cdot}2804 \times \frac{87}{300} \times 0{\cdot}0015 = 101{\cdot}2808$ ft.

$p = 71{\cdot}8607 \times \frac{87}{300} \times 0{\cdot}1041 = 71{\cdot}8305$ ft.

The approximate great circle will appear as a curved line, showing an increase of azimuth from A at A to $A + \delta A$ at B, the average azimuth being $A + \frac{1}{2}\delta A$.

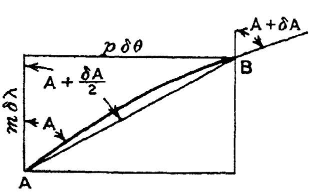


FIG. 164.

Hence

$$\tan(A + \tfrac{1}{2}\delta A) = \frac{p \, . \, \delta\theta}{m \, . \, \delta\lambda}$$

$$= \frac{71{\cdot}8305 \times 2670}{101{\cdot}2808 \times 1110} = 1{\cdot}705966.$$

Whence the distance $AB = m\,.\,\delta\lambda\sec(A$

$$=101{\cdot}2808 \times 1110 \times 1{\cdot}977468 = 222{,}310{\cdot}30\text{ ft.}$$

Incidentally, the convergence $\delta A = \delta\theta \sin\lambda$

$$=2670 \sin 45^\circ\,1'\,27'' = 31'\,28{\cdot}77''.$$

Example†. The following true bearings and distances in statute miles were observed in running a route survey from a base station A to a station D, the latitude and longitude of A being respectively 54° 10′ N. and 72° 12′ E.

AB : N. 80° E., 15 miles ; BC : N. 75° E., 25 miles ; CD : N. 60° E., 20 miles.

Determine the latitude and longitude of D by account, given that for the spheroid 1′ of longitude is 1·15287 miles at the equator, and that 1′ of meridian is $1{\cdot}15090(1-0{\cdot}005\cos 2\lambda)$ miles, λ being the mean latitude.

Lat. diff. between A and D,

$$l = 15\cos 80^\circ + 25\cos 75^\circ + 20\sin 60^\circ = 19{\cdot}075198\text{ mls.}$$

Dep. diff. between A and D,

$$d = 15\sin 80^\circ + 25\sin 75^\circ + 20\sin 60^\circ = 56{\cdot}240770\text{ mls.}$$

Approx. lat. diff. $d\lambda =$ $\qquad = 16{\cdot}574'$.

Mean lat. $= 54^\circ\,18'\,17{\cdot}22''$.

$m = 1{\cdot}15090(1 + 0{\cdot}005\cos 71^\circ\,23'\,25{\cdot}78'') = 1{\cdot}1527364$ mls., where m is the *exact* value of 1′ of meridian.

$$\text{Exact lat. diff.} = \frac{19{\cdot}075198}{1{\cdot}152736} = 16'\,32{\cdot}68''.$$

$$\text{Long. diff.} = \frac{d\,.\sec(54^\circ\,18'\,16{\cdot}34'')}{1{\cdot}15287} = 1^\circ\,23'\,36{\cdot}47''.$$

Lat. D, N. 54° 26′ 32·68″. Long. D, 73° 35′ 36·47″ E.

QUESTIONS ON ARTICLE 6

1†. The following results were obtained in running a traverse survey for a proposed railway :

Station :	A		B		C		D
Deflection Angle :			5° R.		20° L.		
Length (miles) :		10		20		15	

The latitude of A was 57° N. The azimuth of AB (by astronomical observation) 265° E. of N.

Calculate the correction which must be applied to the bearing at D (as obtained from the traverse) to allow for the convergence of the meridians.

Take 69·2 miles = 1° at the centre of the earth. (U.L.)

[58′ 41″]

2†. Assuming a formula of the type

$$\tan \tfrac{1}{2}(A+B)=\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}C,$$

show that the change of azimuth in a long survey line is the product of the difference of longitude at its ends and the sine of the average latitude of its ends.

Determine the approximate increase in azimuth in a traverse which has total northings and eastings each of 42,500 ft. from a station in lat. 59° N., given that 1′ of latitude represents 6092 ft. and 1′ of longitude 3128 ft. in lat. 59° 10′, decreasing thence at the rate of 1·6 ft. per minute. (U.L.)

[11′ 41·6″]

3†. Submit the calculations relative to interpolating three posts at intervals of 5 statute miles between monuments 20 statute miles apart on the 50° 30′ parallel of latitude.

The earth's mean radius may be taken as 20,890,172 ft. (U.L.)

4†. In a network of major triangulation a station A is in lat. 24° 40′ 15″ N. and long. 122° 15′ 35″ E., and an adjacent station B is in lat. 25° 12′ 25″ N. and long. 122° 35′ 40″ E. Calculate the length of the line AB on the assumption that the earth is a sphere with a radius of 20,890,000 ft.

[$\log R = 7{\cdot}3199384$]

Also if A is a ground station on a knoll, 510 ft. in elevation, and the elevation of B is 674 ft., also above mean sea-level, determine the necessary height of the signal at B, assuming that no higher ground intervenes between the stations. (U.L.)

[341,509 ft., or 64·6798 mls. Scaffold at B 15 ft. high approx. with signal 7 to 10 ft. higher.]

5†. In what circumstances will the latitude of the vertex of a great circle course (*a*) exceed, (*b*) equal, the latitude of the higher latitude terminal point?

Illustrate your argument by the determination, *either* graphically *or* by computation, of the latitude of the vertex of the following courses :

(i) from 40° N., 40° W. to 70° N., 20° E.
(ii) from 40° N., 40° W. to 70° N., 120° E. (U.L., Cart.)

[N. 70° 22′ ; N. 85° 18′]

6. If two places A and B lie on different meridians show that the convergence angle increases in value as the latitudes approach 90° and tends to the limiting value of the difference in longitude between the two places.

If the latitude of $A = 55°$ N., the latitude of $B = 60°$ N., and the difference in longitude $= 10′$, what is the convergence?

Check your result by finding the approximate spherical excess of the spherical triangle APB, where P is the pole.

Assume the diameter of the earth $= 7{,}913$ miles. [U.B.]

[8′ 26″]

7. Obtain the latitude and longitude of a point B and the convergence of the meridians through A and B from the data given below, also write down the true azimuth of the line BA.

Latitude of A is 55° N., and longitude of A is 25° E.

Length of AB in arc is 20 min. and the azimuth of AB from the north clockwise and referred to the meridian through A is 70°.

Assuming that the radius of the earth is 3,956 miles, what is the length of AB in miles?

To obtain the latitude of B, use the following equation:

$$\tan \tfrac{1}{2}BP = \frac{\sin \frac{1}{2}(B+P)}{\sin \frac{1}{2}(B-P)} \tan \tfrac{1}{2}\,(\text{Colat. of } A - c). \qquad \text{(U.B.)}$$

[N. 55° 05′ 55·4″; 25° 32′ 50·9″ E.; convergence 23′ 32·2″; azimuth, BA, 250° 23′ 32·2″.]

8. A radio station is situated in an unsurveyed desert at lat. 40° N., long. 80° E. (to the nearest degree).

You are asked to give the latitude and longitude of the station mast to within ¼ mile. The result is required within a day or two. Describe in detail the procedure you would employ for finding:

(a) The latitude.
(b) The longitude.

In each case mention all the instruments and books you use, and explain how you obtain the required degree of accuracy, stating possible causes of error. (T.C.C.E.)

ARTICLE 7: GEODETIC CALCULATIONS

Calculation of spherical triangles. Three methods may be employed in calculating the lengths of the sides in a system of spherical triangles: (1) spherical trigonometry; (2) Delambre's Method; (3) Legendre's Method.

Most of the triangles of the Ordnance Survey were computed by Delambre's Method and checked by Legendre's Method, while all three methods were used in the calculations in connection with the meridional arc which formed the basis of the metric system.

(1) *Spherical trigonometry.* Here the usual problem consists in determining two sides, given one side and three angles. These data avoid the ambiguous case, allowing the application of the Sine Rule; namely,

$$\sin b = \frac{\sin B}{\sin A} \cdot \sin a \quad \text{(see p. 264).}$$

Logarithms should be used methodically in conjunction with an appropriate tabular form. Usually the work is tedious, and the other methods give equally correct results with less labour generally, the differences in the results being but a small fraction in 100 miles.

(2) *Delambre's method.* In this method the angular points are assumed to be joined by straight lines, which, being chords of the arcs, form plane triangles, the spherical angles being reduced to plane angles. The given spherical side is reduced to its chord, and with this and the plane angles the other chords are calculated by plane trigonometry and are then reduced to their corresponding arcs. As a rule, the following method is preferred, being simpler and more expeditious.

(3) *Legendre's method.* This is based upon the theorem that when the sides of a spherical triangle are very small in comparison with the radius of the sphere, each of the angles may be diminished by one-third of the true spherical excess, and the sines of these angles will be proportional to the lengths of the opposite sides, the triangle being regarded as though it were plane.

General procedure. Let A_0, B_0, C_0 be the mean observed values of the spherical angles, A, B, C the corresponding corrected values, and A', B', C' the corresponding plane angles, ϵ'' being the spherical excess in seconds :

(1) Ascertain the total discrepancy δ from 180°, δ being actually ϵ where e is the total error in a plane triangle.

(*a*) If the angles are of equal weight, $\frac{1}{3}$ of δ should be applied to each angle for the final value of the plane angles in Delambre's and Legendre's methods, the calculations for spherical excess not being involved. This follows from the fact that $A' = A - \epsilon/3 = (A_0 - e/3) - \epsilon/3 = A_0 - \delta/3$. The plane angles thus determined are also used in calculating the spherical excess, which is *invariably* involved in the direct method.

(*b*) If the angles are not of equal weight, or if spherical trigonometry is employed, calculate the spherical excess as follows :

(2) Determine the area S of the triangle as though it were plane, using the corrected plane angles of (*a*) in $S = \frac{1}{2} a \,.\, b \,.\, \sin C$, or

$$S = \tfrac{1}{2} \,.\, a^2 \frac{\sin B \,.\, \sin C}{\sin (B + C)} \,.$$

Calculate the spherical excess, using this value of the area in

$$'' \quad S \quad \frac{648000}{\pi R^2} \text{ (with } S \text{ in sq. ft. and } R \text{ in ft.).}$$

This will be the final value in the direct method, not only for angles of equal weight but also for varying weights under a wide range of conditions, since the angle error will have little effect upon ϵ''. It will, how-

ever, serve as a first approximation when conditions demand extreme accuracy, requiring that plane angles be calculated from

$$A' = (A_0 - e_A) - \epsilon/3, \text{ etc.,}$$

where e_A, e_B, e_C are the corrections found by the methods of p. 418. With these values of A', B', and C', a more exact value of ϵ may be found.

(3) Ascertain the total error e in the measurement of the angles, which is the difference between the sum of the observed angles and $180° + \epsilon''$.

Correct the angles by distributing the error among them in accordance with the weight, etc., and, if necessary, from each corrected angle deduct $\frac{1}{3}\epsilon''$.

(4) Calculate the sides by the method prescribed; namely:

Spherical trigonometry. Reduce the length of the side a to its central angle $\bar{a} = \dfrac{180}{\pi R} a$, substitute in $\sin \bar{b} = \dfrac{\sin B}{\sin A} \cdot \sin \bar{a}$, etc., and finally express as lengths $b = \dfrac{\pi R}{180} \bar{b}$, etc.

Delambre's method. Reduce the given length a, say, to its central angle $\bar{a}$, calculate the corresponding chord from $\text{ch}\, a = 2R \sin \frac{1}{2}\bar{a}$, and with the given plane angles, A', B', C', determine the remaining chords from

$$\frac{\text{ch}\, b}{\sin B} = \frac{\text{ch}\, c}{\sin C} = \frac{\text{ch}\, a}{\sin A}.$$

Reduce these chords to central angles, $\bar{b}$, $\bar{c}$, by the relation $\sin \frac{1}{2}\bar{b} = \dfrac{\text{ch}\, b}{2R}$, and finally determine the arcs b, c, from $b = \dfrac{\pi R}{180°} \bar{b}$, etc., or appropriate tables.

Legendre's method. Using the corrected and diminished angles and the given side, calculate the remaining sides from $b = \dfrac{\sin B}{\sin A} a$, etc.

The following example embodies the procedure and calculations involved in applying the foregoing methods.

Example. The following are the mean observed angles in a spherical triangle ABC, the relative weights being respectively 6, 5, and 4:

$$A_0 = 62° \, 27' \, 28{\cdot}82''\,; \quad B_0 = 55° \, 45' \, 16{\cdot}49''\,; \quad C_0 = 61° \, 47' \, 20{\cdot}18''.$$

The length of the side $BC = a$ is 105042·08 ft.

Compute the lengths of the sides AB and CA on the assumption that the mean value of a minute of arc is 6076·90 ft.

Submit your results in tabular form.

Spherical excess. Deducting $\frac{1}{3}$ of the total discrepancy $\delta = 5{\cdot}49''$ from each angle, the approximate plane angles are accordingly :

$$A' = 62^\circ\ 27'\ 26{\cdot}99'' ;\ B' = 55^\circ\ 45'\ 14{\cdot}66'' ;\ C' = 61^\circ\ 47'\ 18{\cdot}35''.$$

Calculating the value of ϵ'' from $S \times \dfrac{648000}{\pi R^2}$ with $S = \dfrac{\frac{1}{2}a^2 \sin B \, . \sin C}{\sin (B+C)}$.

$$\begin{aligned}
\log \tfrac{1}{2} &= \bar{1}{\cdot}6989700 \\
2 \log a &= 10{\cdot}0427266 \\
\log \sin B &= \bar{1}{\cdot}9173110 \\
\log \sin C &= \underline{\bar{1}{\cdot}9450793} \\
& \ 9{\cdot}6040869 \\
& \ \underline{\bar{1}{\cdot}9477610} \ \log \sin (B+C). \\
\log S &= 9{\cdot}6563259
\end{aligned}$$

$\log \dfrac{648000}{\pi R^2}$ with a mean value of R of

$$\begin{aligned}
20{,}888{,}629 \text{ ft.} &= \overline{10}{\cdot}6746069 \\
\log S &= \underline{9{\cdot}6563259} \\
\text{and } \log \epsilon'' &= 0{\cdot}3309328
\end{aligned}$$

$\epsilon'' = 2{\cdot}143''$, leaving the error e in the angles, $5{\cdot}49'' - 2{\cdot}14'' = 3{\cdot}35''$.

Distributing the error in the angles in accordance with the weights :

$$\tfrac{1}{6} \qquad \frac{0{\cdot}166}{0{\cdot}616} \times 3{\cdot}35 = 0{\cdot}90''.$$

$$e_B = \qquad \frac{0{\cdot}30}{0{\cdot}616} \times 3{\cdot}35 = 1{\cdot}09''.$$

$$e_C = \qquad \frac{0{\cdot}25}{0{\cdot}616} \times 3{\cdot}35 = 1{\cdot}36''.$$

Substituting these for the corrected spherical angles :

$$A = 62^\circ\ 27'\ 27{\cdot}92'' ;\ B = 55^\circ\ 45'\ 15{\cdot}40'' ;\ C = 61^\circ\ 47'\ 18{\cdot}82''.$$

Further, subtracting $\frac{1}{3}\epsilon$ for the final plane angles :

$$A' = 62^\circ\ 27'\ 27{\cdot}21'' ;\ B' = 55^\circ\ 45'\ 14{\cdot}69'' ;\ C' = 61^\circ\ 47'\ 18{\cdot}11''.$$

(1) *Spherical trigonometry.* The side a has an angular value of $17'\ 17{\cdot}12''$.

$$\begin{aligned}
\log \sin \bar{a} &= \bar{3}{\cdot}7012562 \\
\log \sin B &= \underline{\bar{1}{\cdot}9173121} \\
& \ \bar{3}{\cdot}6185683 \\
\log \sin A &= \underline{\bar{1}{\cdot}9477621} \\
\log \sin \bar{b} &= \bar{3}{\cdot}6708062
\end{aligned}
\qquad
\begin{aligned}
\log \sin \bar{a} &= \bar{3}{\cdot}7012562 \\
\log \sin C &= \underline{\bar{1}{\cdot}9450790} \\
& \ \bar{3}{\cdot}6463352 \\
\log \sin A &= \underline{\bar{1}{\cdot}9477621} \\
\log \sin \bar{c} &= \bar{3}{\cdot}6985731
\end{aligned}$$

$\bar{b} = 16{\cdot}11249'$ (by interpolation.) $\bar{c} = 17{\cdot}17724'$.

$b = 16{\cdot}10941'$ (exactly). $\bar{c} = 17{\cdot}17310'$.

{*Ex. Chambers' Tables for small angles* (*working omitted*)}

$$b = 16{\cdot}10941 \times 6076{\cdot}9 = 97895{\cdot}70 \text{ ft.} ;$$

$$c = 17{\cdot}17310 \times 6076{\cdot}9 = 104359{\cdot}02 \text{ ft.}$$

(2) *Delambre's method.*

Chord $a = 2R \sin \tfrac{1}{2}\bar{a} = 2 \times 20{,}888{,}692 \sin \tfrac{1}{2}(17\cdot28534)'$.

$$\begin{aligned}
\log R &= 7\cdot3199099\\
\log 2 &= 0\cdot3010300\\
\log \sin \tfrac{1}{2}\bar{a} &= \bar{3}\cdot4003736\\
\hline
\log \text{ch } a &= 5\cdot0213135\\
\log \sin B &= \bar{1}\cdot9173111\\
\hline
& \ 4\cdot9386246\\
\log \sin A &= \bar{1}\cdot9477613\\
\hline
\log \text{ch } b &= 4\cdot9908633\\
\log 2R &= 7\cdot6209399\\
\log \sin \tfrac{1}{2}\bar{b} &= 3\cdot3699234\\
\tfrac{1}{2}\bar{b} &= 8\cdot05745'\\
\bar{b} &= 16\cdot11490'
\end{aligned}$$

Using ch a and the corrected plane angles in $b = \dfrac{\sin B}{\sin A}\, a$ and $c = \dfrac{\sin C}{\sin A}\, a$:

$$\begin{aligned}
\text{ch } a &= 104864\cdot78 \text{ ft.}\\
\hline
\log \text{ch } a &= 5\cdot0213135\\
\log \sin C &= \bar{1}\cdot9450782\\
\hline
& \ 4\cdot9663917\\
\log \sin A &= \bar{1}\cdot9477613\\
\hline
\log \text{ch } c &= 5\cdot0186304\\
\log 2R &= 7\cdot6209399\\
\hline
\log \sin \tfrac{1}{2}\bar{c} &= \bar{3}\cdot3976905\\
\tfrac{1}{2}\bar{c} &= 8\cdot58944'\\
\bar{c} &= 17\cdot17888'
\end{aligned}$$

{*Ex. Chambers' Tables for small angles* (see p. 320)}

$b = \bar{b} \times 6076\cdot9 = 97928\cdot65$ ft., since $\bar{b} = 2 \sin^{-1} b/2R$.

$c = \bar{c} \times 6076\cdot9' = 104393\cdot13$ ft. since $\bar{c} = 2 \sin^{-1} c/2R$.

(3) *Legendre's method.* Using the reduced angles and given side in the Sine rule for plane triangles:

$$\begin{aligned}
\log \text{side } a &= 5\cdot0213633\\
\log \sin B &= \bar{1}\cdot9173111\\
\hline
& \ 4\cdot9386744\\
\log \sin B &= \bar{1}\cdot9477613\\
\hline
\log \text{side } b &= 4\cdot9909131\\
b &= 97929\cdot40 \text{ ft.}
\end{aligned}$$

$$\begin{aligned}
\log \text{side } a &= 5\cdot0213633\\
\log \sin C &= \bar{1}\cdot9450782\\
\hline
& \ 4\cdot9664415\\
\log \sin A &= \bar{1}\cdot9477613\\
\hline
\log \text{side } c &= 5\cdot0186802\\
c &= 104395\cdot12 \text{ ft.}
\end{aligned}$$

	w	Mean spherical angles		Subtending side (min.)
		Observed	Corrected	
A	6	62° 27′ 28·82″	62° 27′ 27·92″	17·28534
B	5	55° 45′ 16·49″	55° 45′ 15·40″	16·11249
C	4	61° 47′ 20·18″	61° 47′ 18·82″	17·17724
		180° 00′ 5·49″	180° 00′ 2·14″	

	w	Corrected plane angles	Geodetic lengths (ft.)		
			Direct	Delambre	Legendre
A	6	62° 27′ 27·21″	105042·08	105042·08	105042·08
B	5	55° 45′ 14·69″	97895·70	97928·65	97929·40
C	4	61° 47′ 18·11″	104359·02	104393·13	104395·12
		180° 00′ 0·01″			

QUESTIONS ON ARTICLE 7

1†. In one of the triangles of a major triangulation, a side a is 124,632·40 ft. and the mean observed angles are of equal weight, being $A = 72° \ 24' \ 12·4''$; $B = 58° \ 19' \ 33·3''$; and $C = 49° \ 16' \ 17·9''$.

Compute the lengths of the sides by Legendre's method, assuming the mean radius of the earth to be 20,888,630 ft. (U.L.)

[$b = 111{,}274·74$ ft., $c = 99{,}083·80$ ft.]

2†. In one of the triangles of a major triangulation, a side a is 124,632·40 ft., and the mean angles are of equal weight, being

$$A = 72° \ 24' \ 12·4'', \ B = 58° \ 19' \ 33·3'', \text{ and } C = 49° \ 16' \ 17·9''.$$

Compute the lengths of the remaining sides by Delambre's method, assuming the mean radius of the earth to be 20,888,630 ft. (U.L.)

[Since in each case the angles are of equal weight, spherical excess is not involved, and 1·2″ is merely deducted from each angle. $b = 111{,}260·25$ ft., $c = 99{,}087·84$ ft. by interpolation.]

3†. In one of the triangles of a major triangulation, a side a is 124,632·40 ft., and the mean angles are of equal weight, being

$$A = 72° \ 24' \ 12·4'', \ B = 58° \ 19' \ 33·3'', \text{ and } C = 49° \ 16' \ 17·9''.$$

Compute the lengths of the remaining sides by spherical trigonometry, assuming the mean radius of the earth to be 20,888,630 ft., and determine the error in area that would result if ABC were regarded as a plane triangle. (U.L.)

[By interpolation $b = 111{,}274·00$ ft.; $c = 99{,}093·32$ ft.; 66,000 sq. ft.]

4†. In one of the triangles of a major triangulation, a side a is 130,252·8 ft., and the mean observed angles are of equal weight, being

$$A = 64° \ 16' \ 10·2''; \ B = 55° \ 29' \ 35·2'' \text{ and } C = 60° \ 14' \ 18·5''.$$

Compute the lengths of the sides by Legendre's method, assuming the mean radius of the earth to be 20,890,172 ft. [log $R = 7·3199422$.] (U.L.)

[$b = 119{,}149·9$ ft.; $c = 125{,}517·9$ ft.]

SECTION VI

ERRORS OF SURVEYING

INTRODUCTION

Surveying may be defined from another aspect as the science of the control, assessment, and distribution of errors of observation, the subject holding a unique place in the application of the theory of errors. For this reason all those methods which involve consideration of errors are treated in this section.

Error and discrepancy. The difference between any measurement of a quantity and the true value of that quantity is the *true* error of measurement. But since the *true* value of a measured quantity is never known, the *true* error of measurement is never known.

This statement must not be confused with cases where a number of measurements should fulfil known conditions, as for example, the interior angles of a polygon should sum up to $(2N-4)$ right angles ; but here the true error in the *sum* of the measurements is known and not the error in any *single* measurement.

A discrepancy is the observed difference between two like measurements, each of which contains an error that may or may not be appreciable.

(*a*) *A discrepancy is not an error* ; (*b*) *a large discrepancy indicates a mistake in the observations* ; *and* (*c*) *a small discrepancy between two measurements is no criterion that the error is small.*

Nature of errors. Errors of measurement are of three kinds : (i) mistakes ; (ii) systematic errors ; and (iii) accidental errors.

(i) *Mistakes* are errors which originate in the mind of the observer, and arise from carelessness, inexperience and mental confusion.

(ii) *Systematic errors* arise from known sources, and can be eliminated, as in the cases of incorrect length or stadia interval, imperfect adjustments, etc.

(iii) *Accidental errors* are those which remain after mistakes and systematic errors have been eliminated. These are due to imperfections of human sight and touch, imperceptible changes in the instruments, indeterminate variations of temperature, pull, etc., and other sources of systematic error.

Whenever the value of a quantity is found by adding together the measurements of its several constituent quantities, any source of accidental error becomes a source of **compensating error**, since the sign is as

likely to be plus as minus in each of the several measurements ; but any source of systematic error becomes a source of **cumulative error**, since *under certain conditions*, systematic errors have always the same sign, though as implied the net effect may be conditional. Cumulative errors from any one source affect the total result in the same way, whereas compensating errors tend to balance one another.

Sign. If the measurement of a quantity is *greater* than the true value of that quantity, the error is *plus* ; if *less* than the true value, the error is *minus*.

(*a*) The total error is not due to any one cause : it is the algebraical sum of errors due to different causes.

(*b*) An accidental error is as likely to be plus as minus, but a systematic error under the same conditions has always the same sign and the same magnitude.

(*c*) Systematic errors remain unchanged both in magnitude and sign when the measurements are repeated under precisely the same conditions. Nor are they affected or reduced by taking the mean of all the measurements. On the other hand, the accidental error of the mean is likely to be less than the accidental error of a single measurement ; and, in theory, that portion of the total error accruing from accidental errors would absolutely disappear from the means of an indefinitely large series of observations made under precisely the same conditions.

(i) Mistakes are detected by checking results with existing data and known conditions.

(ii) Systematic errors are avoided or eliminated, partly by systematic operations and partly by calculation and correction.

(iii) Accidental errors cannot be eliminated ; but experience has shown that they follow certain mathematical laws which are fundamental in the *Theory of Least Squares* ; namely, (1) small errors occur more frequently than large ones ; (2) positive and negative errors are equally numerous ; (3) very large errors never occur.

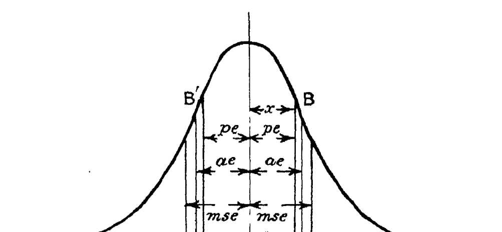


Fig. 165.
Probability Curve.

The criteria of error for a single measurement are (1) *true probable error* (p.e.) $\pm 0{\cdot}6745 \sqrt{\dfrac{\Sigma x^2}{n}}$; (2) *true mean square error* (m.s.e.) $\pm \sqrt{\dfrac{\Sigma x^2}{n}}$; and (3) average error $\pm \dfrac{\Sigma x}{n}$, where n is the number of observations and x the *true* error of any individual measurement. Mean square and average errors are only of secondary importance in modern surveying.

ARTICLE 1: ERRORS OF SURVEYING

Probable error. The probable error of a measured quantity is a magnitude such that the chances are *even* that the true error contained in that quantity is greater or less than the probable error. It is therefore the limit within which the probability is one-half (1/2) that the truth will fall, since a probability expressed by 1/1 indicates a certainty, a probability of 0/1 an impossibility, and generally $1/n$ that the event will occur on the average once out of n times.

For example, if $196{\cdot}43 \pm 0{\cdot}16$ is the mean of a number of observations, the true value is as likely to be between 196·27 and 196·59 as it is to be some value greater or less.

A curve expressing the probability y that an error of given magnitude x will occur in an infinitely large number of measurements is shown in Fig. 165, the exact shape depending upon a constant k determined primarily by precision. This represents the probability curve:

$$y = \frac{k}{\sqrt{\pi}} e^{-k^2x^2}\, dx, \quad \text{......}(1)$$

where e is the base of Naperian logarithms, namely, 2·71828.

Now if the total number of observations be n, then the number of these with errors between x and $x + dx$ is

$$\frac{nk}{\sqrt{\pi}} e^{-k^2x^2}\, dx, \quad \text{......}(2)$$

and this group contributed to the sum Σx_r^2, a total of

$$\frac{nk}{\sqrt{\pi}} e^{-k^2x^2}\, x^2\, dx,$$

where Σx_r^2 represents the sums of the squares of all the errors from the mean.

Hence the value of $\sigma^2 = \frac{\Sigma x_r^2}{n}$ for *all* observations is

$$\frac{nk}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-k^2x^2} x^2\, dx \cdot \frac{1}{n} = \frac{2k}{\sqrt{\pi}} e^{-k^2x^2} x^2\, dx = \sigma^2, \quad \ldots\ldots\ldots\ldots\ldots(3)$$

where σ is the *theoretical standard error.*

Equation (3) may be evaluated by integrating in parts, putting $kx = z$; and thus it follows that

$$\sigma^2 = \frac{\Sigma x_r^2}{n} = \frac{1}{2k^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The probable error x is obviously given by symmetrical values of $\pm x$,

or
$$\frac{k}{\sqrt{\pi}} \int e^{-k^2x^2}\, dx = \tfrac{1}{2}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

or
$$\frac{2k}{\sqrt{\pi}} \int_0 e^{-k^2x^2}\, dx = \tfrac{1}{2}; \quad \ldots\ldots\ldots(5a)$$

and again putting $kx = z$, $dx = \frac{1}{k}\, dz$;

that is,
$$\int_0^{kx} e^{-z^2}\, dz = \tfrac{1}{4}\sqrt{\pi}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Now the values of $\int_0^z e^{-z^2}\, dz$ have been tabulated, and they can be approximated to by expressing the integral as a continued fraction. There is, however, no set evaluation of this integral in terms of known functions of z, but the values have been tabulated for different values of z by the various developments of the continued fraction method given by Lagrange.

By interpolation it is found that

$$\int_0^{0\cdot67448975/\sqrt{2}} e^{-z^2}\, dz = \tfrac{1}{4}\sqrt{\pi},$$

where 0·67448... is merely a number derived from the tabulated values of $\int_0^z e^{-z^2}\, dz$, and not, as is sometimes thought, a simple function of π.

Hence $kx = \frac{0\cdot6745}{\sqrt{2}}$, where x is the probable error.

But $\sigma^2 = \frac{\Sigma x_r^2}{n} = \frac{1}{2k^2}$; and, since $k = \frac{1}{\sigma\sqrt{2}}$, $kx = \frac{x}{\sigma\sqrt{2}}$.

Hence
$$x = 0\cdot6745\sigma = 0\cdot6745\sqrt{\frac{\Sigma x_r^2}{n}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Now in practice the standard error is calculated empirically from observations with residuals d instead of x_r, and allowance must be made for the fact that the mean used is not the true theoretical mean, the

difference between the arithmetic mean and its individual observed values not being true errors, but apparent errors, or residuals.

But if x be the true error of any individual measurement, d the residual, and δx the difference :

$$d - \delta x = x, \quad \text{and} \quad d^2 - 2d \,.\, \delta x + (\delta x)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now δx is constant for any particular series of measurements, and is the difference between the real and apparent errors in any given case. Then, since the true probable error of a single observation is

$$0{\cdot}6745 \sqrt{\frac{\Sigma x^2}{\quad}}$$

it follows on taking the summation of each of the terms in (i) that

$$\Sigma d^2 + (\delta x)^2 = n\,(\text{p.e.})^2 \ \text{ for } \Sigma d = 0,$$

the values of d being equally likely plus or minus. As an approximation δx may be taken equal to the p.e., since both these values decrease with increase in n. Then $0{\cdot}6745\ \Sigma d^2 + (\text{p.e.})^2 = n(\text{p.e.})^2$. Whence the p.e. of a single observation,

$$E_s = \pm 0{\cdot}6745 \sqrt{\frac{\Sigma d^2}{n-1}}$$

Similarly the apparent m.s.e. may be written

It follows from the theory of least squares that the probable error (m.s.e. or a.e.) of the mean is $1/\sqrt{n}$ of the probable error of a single measurement. Thus the p.e. of the mean,

$$E_m = \frac{E_s}{\sqrt{n}} = \pm 0{\cdot}6745 \sqrt{\frac{\Sigma d^2}{n(n-1)}}\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$E_m{}' = \pm \sqrt{\frac{\Sigma d^2}{n(n-1)}}\,; \quad \ldots\ldots\ldots(10a)$$

and (uniformly) the average error,

$$E_m{}'' = \pm \frac{\Sigma d}{n\sqrt{n-1}} \cdot \quad \ldots\ldots\ldots\ldots(10b)$$

Probable value. The most probable value of a directly measured quantity is the arithmetical mean of all the observations if these are of equal weight, or is the weighted mean if the observations are of unequal weight. Hence if a quantity be observed as A_1 with a weight w_1, and as A_2 with a weight w_2, the most probable value will be :

$$\frac{w_1A_1 + w_2A_2}{w_1 + w_2},$$

and the weight of this value will be $(w_1 + w_2)$.

Thus the most probable value of a base line which was measured under different conditions as 2835·42 m. with weight 3 and 2834·54 with weight 1 is 2835·20 m., as indicated.

$$\begin{array}{r} 3\,(2835{\cdot}42) = 8506{\cdot}26 \\ 1\,(2834{\cdot}54) = 2834{\cdot}54 \\ 4\,\overline{)\,11340{\cdot}80} \\ \hline 2835{\cdot}20 \end{array}$$

Least squares. It can be shown from the probability equation that the most probable values of a series of errors arising from observations of equal weight are those of which the sum of the squares is a minumum. Hence the fundamental law of least squares, which shows for observations of equal weight that the most probable value of an observed quantity is that which makes the sum of the squares of the residual errors Σd^2 a minimum; and it may be deduced likewise that the most probable value of a quantity observed with unequal weights is that which makes the sum of the weighted squares of the residuals, $\Sigma(wd)^2$, a minimum.

The theoretical basis of least squares is the assumption that accidental errors will wholly disappear from the mean of an indefinitely large series of independent, like-conditioned observations of the same quantity; and that, therefore, if systematic errors be eliminated from these observations, the arithmetical mean will be the most probable value of that quantity. These conditions are never realised in practice, for (*a*) an indefinitely large number of observations is not obtainable; (*b*) no two observations can be made under precisely the same conditions; while (*c*) systematic errors can never be completely eliminated. The most probable value, therefore, is strictly a theoretical value, and the more complete the fulfilment of the conditions involved, the nearer that value will approach the ideal value as a limit. This limit is never attained: therefore, however accurate the observations may be, the probable value will always contain an error. On the other hand, great use may be made of the underlying principles, for the practical instinct of the surveyor seeks something beyond the dogmatism of the mathematician, who despising more or less empirical formulae, often visualises a particular adjustment as an insoluble problem in the analysis of errors.

(*a*) **Probable error of a single observation** (E_s) indicates the precision that may be expected in any single observation made under the same conditions. Thus, for example, if the probable error of a single observation in angular measurement is ± 5 seconds, the probable error for any other angle may be expected to be about ± 5 seconds, provided the same observer uses the same instrument under the same conditions.

(*b*) **Probable error of the mean** (E_m) indicates the precision that may be expected in the mean of any series of observations made under the same conditions.

For example, a series of measurements of a base line resulted in a mean of 1265·62 ft. with a probable error in that value of $\pm 0{\cdot}049$ ft. Therefore, for the mean of any other set of remeasurements, a probable

error of $\pm 0{\cdot}049$ ft. may be expected, provided the same conditions prevail and the same party, organisation and instruments are employed in that set of measurements.

The probable errors of different series of measurements are used in comparing the respective degrees of precision of those series. Thus, for example, the above base was remeasured, 1265·65 ft. and $\pm 0{\cdot}063$ ft. resulting as the mean and its probable error. This indicates that the precision of the first set of measurements is to the second as 0·063 is to 0·049, or as 9 : 7.

(*c*) **Relative weight.** Probable error also determines the relative weight that should be given to different sets of observations, since the weights of observations vary inversely as the squares of their probable errors. Thus, in the case of the base line measurements, as cited in the preceding paragraph, the respective weights are as $\frac{1}{(0{\cdot}049)^2}$ to $\frac{1}{(0{\cdot}063)^2}$ or as about 1·65 to 1.

Formulae. In the following summary, E_s and E_m are respectively the probable errors of a single observation and of the mean of all the observations, n being the number of observations and d the residual error; that is, the difference between any one observation and the mean of all the observations:

(*a*) The p.e. of a single observation, $E_s = 0{\cdot}6745\sqrt{\frac{\Sigma d^2}{n-1}}$. ....(8)

(*b*) The p.e. of the mean, $E_m = 0{\cdot}6745\sqrt{\frac{\Sigma d^2}{n(n-1)}} = \frac{E_s}{\sqrt{n}}$. ....(9)

(*c*) The p.e. of the weighted or general mean,

$$E_w = 0{\cdot}6745\sqrt{\frac{\Sigma wd^2}{(n-1)\,\Sigma w}}, \quad \text{......................(10)}$$

where Σw is the sum of the weights.

(*d*) The probable error of a quantity with a weight w is equal to E_w divided by the square root of w.

(*e*) The probable error of Q, the sum or difference of several independent quantities $q_1, q_2, q_3, q_4 \ldots q_n$, the probable errors of which are $e_1, e_2, e_3, e_4 \ldots e_n$ respectively,

$$E_q = \sqrt{e_1^2 + e_2^2 + e_3^2 + e_4^2 + \ldots e_n^2}. \quad \text{.................(11)}$$

(*f*) The probable error of a product Aq, where A is a known quantity, q an observed quantity, and e the probable error of q,

$$E_p^2 = (Ae)^2, \quad \text{or} \quad E_p = Ae. \quad \text{.........................(12)}$$

(*g*) The probable error of P, the product of q_1 and q_2, the probable errors of which are e_1 and e_2 respectively,

$$E_p^2 = q_1^2 \,.\, e_2^2 + q_2^2 \,.\, e_1^2. \quad \text{..........................(13)}$$

Example. Assess the errors that may be expected in ordinary angular measurement with theodolites reading by vernier to 20″.

When a theodolite in perfect adjustment is accurately levelled up, the main sources of accidental error will be (i) reading the vernier or microscopes, and (ii) bisecting the signal or point sighted.

(i) If the least count of a vernier be x, then the maximum possible error of reading is $\frac{1}{2}x$; but there is also the likelihood of the value being anything between 0″ and $\frac{1}{2}x$ and $\frac{1}{2}x$ and x. Consequently the p.e. will be $\pm\frac{1}{4}x$ for a single vernier, which is $\pm 5''$ when the least count is 20″. Embodied with this an allowance may be made for imperfect dividing which may be (say) 1″.

(*a*) Hence p.e. for a single vernier reading is $\pm\alpha = \pm 6''$, with $x = 20''$.

(*b*) Also p.e. when both verniers are read is $\pm\dfrac{\alpha}{\sqrt{2}} = \pm 6''/\sqrt{2}$,, ,,

(*c*) While p.e. when both verniers are read with both faces of the theodolite is $\pm\dfrac{\alpha}{2} = \pm 6''/\sqrt{4} = 3''$,,

But in any angular measurement the readings of (*a*), (*b*), or (*c*) will occur for two sights or pointings :

$$\text{hence p.e. per angle} = (a)\ \pm\sqrt{2}\alpha\ ;\ (b)\ \pm\alpha\ ;\ (c)\ \pm\alpha/\sqrt{2}.$$

(ii) The error of sighting a signal will occur once for each sight in (*a*) and (*b*) and twice for each sight in (*c*). That is, the error in (*a*) and (*b*) will be $\sqrt{2}\beta$ and in (*c*) $\sqrt{2}\beta/\sqrt{2} = \beta$.

The error β will vary with the nature of the signal, the distance, the atmospheric conditions, and the power and quality of the telescope. It may vary from 2″ to 10″ in ordinary work, and may be taken at 6″ on the average with a 5 in. or 6 in. vernier instrument. In addition, there is the error of centring the theodolite over the station, and, although this may be negligible in triangulation, it can be serious when traversing with short lines.

The p.e. from both sources will be the square root of the sum of the squares of the errors from the individual sources :

(a) : $\sqrt{2\alpha^2 + 2\beta^2} = 12''$; (b) : $\sqrt{\alpha^2 + 2\beta^2} = 10{\cdot}4''$; (c) :

*Example**. In order to investigate the precision of chaining, a line nominally 1,500 ft. in length was measured with the band chain under the same conditions, the discrepancies from 1,500 ft. being as follows :

$$-{\cdot}42,\quad +{\cdot}12,\quad +{\cdot}66,\quad -{\cdot}48,\quad -{\cdot}10,\quad 0,\quad +{\cdot}24,\quad -{\cdot}56,\quad +{\cdot}62,\quad -{\cdot}18.$$

Determine the most probable length of the line and the probable error of a single measurement. State also if you consider the square root law is justified, and, if so, the value of the coefficient per 100 ft. unit. (U.L.)

Discrepancies $d_1, d_2, \ldots d_{10}$ from mean length of 1499·99 ft. are in order: $-\cdot 41$, $+\cdot 13$, $+\cdot 67$, $-\cdot 47$, $-\cdot 09$, $+\cdot 01$, $+\cdot 25$, $-\cdot 55$, $+\cdot 63$, $-\cdot 17$, while the corresponding values of the squares $d_1^2 \ldots d_{10}^2$ are respectively 16·81, 1·69, 44·89, 22·09, 0·81, 0·01, 6·25, 30·25, 38·69, and $2{\cdot}89 \times 10^{-2}$, giving $\Sigma d^2 = 1{\cdot}6538$.

Applying the formula, $E_s = \pm 0{\cdot}6745 \sqrt{\dfrac{\Sigma d^2}{n-1}}$, $E_s = \pm 0{\cdot}288$ ft.,

and, assuming the square root law, $E_s = c\sqrt{L}$ with L in 100 ft. units, $c = 0{\cdot}075$, and $E_s = 0{\cdot}075\sqrt{L}$ ft.

QUESTIONS ON ARTICLE 1

1*. The following *ten* separate measurements of an angle were taken on the *A* vernier of a 5-inch theodolite (No. I), the observed angle being 54° plus the following values in minutes:

$$26\tfrac{1}{3},\ 28,\ 27\tfrac{2}{3},\ 25\tfrac{2}{3},\ 28\tfrac{1}{3},\ 26\tfrac{2}{3},\ 26\tfrac{1}{3},\ 27,\ 27\tfrac{2}{3},\ 26\tfrac{1}{3}.$$

Determine the probable error of a single measurement.

If the above instrument is to be used conjointly on a compound traverse with two other theodolites (Nos. II and III) which show respective probable errors of 0·4′ and 0·3′ in a single measurement, state the relative weights you would apply to observations made with the three instruments. (U.L.)

[0·515′; 1 : 1·65 : 2·95]

2†. The following data were obtained in testing a stadia telescope fitted with a focusing tube, the telescope being level during the tests.

Distance, D - - -	100	200	300	400	500 ft.
Object glass to axis, c -	4·85	4·82	4·80	4·78	4·77 in.
Staff intercept - -	0·97	1·99	2·98	3·97	5·00 ft.

Focal length of object glass, 10 in.

Determine the mean value of the interval factor (specified 100), and state the probable error of this value, assuming that you are permitted to apply the theory to so few observations. What error ratio would be introduced by accepting the multiplier as specified?

[With $f+c = 1{\cdot}24$ ft., the mean value of the multiplier is $100{\cdot}43 \pm 0{\cdot}247$, the latter fraction being the p.e. of the mean. Error ratio, 1 : 233.]

3. Explain what is meant by the probable error of a measurement, and state what you consider to be suitable values for the probable error of the following operations when conducted with the utmost refinement:

(*a*) The measurement of a primary base line, 5 miles long, by invar wires or tapes in catenary.

(*b*) The measurement of each horizontal angle in geodetic triangulation.

(*c*) The running of a line of precise level, 50 miles long.

Selecting one of these items, give a list of the individual errors which contribute to the total probable error. (I.C.E.)

4. (*a*) Enumerate the principle of least squares. Show how this principle is used for determining two unknowns in linear equations. Find expressions for the unknowns in terms of the coefficients and measured quantities.

(*b*) Find the normal equations from the following observations :

$1{\cdot}2x + 4{\cdot}3y$ was measured as $+15{\cdot}0$.
$-2{\cdot}4x + 1{\cdot}0y$ was measured as $-2{\cdot}2$.
$-1{\cdot}5x - 0{\cdot}7y$ was measured as $-4{\cdot}7$.
$2{\cdot}3x - 2{\cdot}0y$ was measured as $-1{\cdot}4$. (U.C.T.)

[$+14{\cdot}74x - 0{\cdot}79y = +27{\cdot}11$,
$-0{\cdot}79x + 23{\cdot}98y = +68{\cdot}39$.]

5. Define mean square error, residual error, and weight, and from your definitions deduce the mean square error m of an observation as deduced from a series of observations on a single quantity as

$$m = \sqrt{\frac{(vv)}{n-1}},$$

where v is the residual error and n the number of observations.

Find the best value of the angle ABC, its mean square error and the weight of a single observation if the weight 1 corresponds to a m.s.e. of $\pm 5''$.

Observed values of ABC :	32° 34′ 07″	32° 34′ 09″
	32° 34′ 05″	32° 34′ 06″
	32° 33′ 58″	32° 34′ 00″
	32° 34′ 03″	32° 34′ 04″

(U.C.T.)

[32° 34′ 04·0″; $\pm 3{\cdot}265''$; 1·90.]

6. (*a*) Calculate the m.s.e. of the height difference h between two points if

$$h = a \tan \alpha + \frac{1-k}{2R} a ,$$

if a, the distance = 12,424 ft. ± 2 ft. ;
α the angle of elevation = 4° 33′ 20″ $\pm 5''$;
k is a coefficient = $0{\cdot}13 \pm 0{\cdot}03$;
R is the radius of the earth = 4,000 miles.

(*b*) A surveyor A observes an angle 12 times and finds the mean value to be 47° $\pm$14′ 27·2″ and the m.s.e. of a single observation $\pm 2{\cdot}5''$. B derives as the mean of 20 observations 47° 14′ 24·0″, the m.s.e. of a single observation being 1·8″.

Calculate the final value of the angle and its m.s.e. (U.C.T.)

[(*a*) 0·377 ft. ; (*b*) 47° 14′ 25·1″, $\pm 1{\cdot}5''$.]

ARTICLE 2 : NORMAL EQUATIONS AND CORRELATES

Observation equations. An observation equation is the symbolic equality of a quantity and its observed value ; as $A = 32°\ 16'\ 20''$ or $A + B = 82°\ 32'\ 12''$. Such a quantity is *directly observed* when its magnitude is tacitly expressed as A, or by any single measurement ; and is

indirectly observed when simultaneous observations or conditions will be involved, as in the case of $A+B$. A *conditioned quantity* is one of a set of quantities that must fulfil rigorous geometrical conditions, and its most probable value is influenced by the observation of other quantities, as in the case of $A+B+C=180°$. When several conditions must be fulfilled, the most probable values are usually obtained by means of undetermined multipliers known as **correlates**.

(*a*) **Directly-observed independent quantities.** It has already been stated that the most probable value of a directly-measured quantity is the arithmetical mean of the observations if these are of equal weight, and is the weighted mean if the observations are of unequal weight (p. 410).

(*b*) **Indirectly-observed independent quantities.** The general case of indirectly-observed independent quantities introduces *normal equations*, which are derived from the relevant observation equations. A normal equation is determined for each quantity for which the most probable value is required, and the individual values are obtained by simultaneous solution.

Normal equations. Consider a round of angles observed at a central station, the horizon closing with three angles x, y, and z, which are geometrically fixed by the condition $x+y+z=360°=-d$ (say).

(i) If the angles are measured once or are the means of equal numbers of equally precise measurements, the error e in the round is $(x+y+z+d)$, and the most probable value of each of the angles will follow if $\frac{1}{3}e$ be applied to each of the observed values.

(ii) If, however, one angle is measured directly and the others indirectly by subtraction from observed values of two or three angles together, or in some way that the conditions of measurement are varied, the error equation takes the form $(ax+by+cz+d)$; and if the measurements are repeated, giving separate observations, x_1, y_1, z_1, x_2, y_2, z_2, etc., the errors will be $e_1=(ax_1+by_1+cz_1+d)$; $e_2=(ax_2+by_2+cz_2+d)$, etc. Now the theory of least squares requires that $\Sigma(e)^2=\Sigma(ax+by+cz+d)^2$ shall be a minimum, and if the last expression be differentiated in order with respect to x, y, and z, and equated to zero, it follows that:

$$\Sigma a(ax+by+cz+d)=0\ ;\quad \Sigma b(ax+by+cz+d)=0\ ;\quad \Sigma c(ax+by+cz+d)=0,$$

being the fundamental equation multiplied by the coefficient of x, y, and z respectively.

These are known as **normal equations**, and the solution of such will lead to the most probable values of x, y, and z.

(iii) Further, if the precision of measurement be varied among the angles, either by the use of different instruments, or by methods reiterating the measurements, the observations will be accorded weights w_x, w_y, and w_z which follow from the probable errors of individual measurement.

The error equation is now further complicated in that the expression

$$\Sigma(w_x ax+w_y by+w_z cz+d)^2$$

must be a minimum ; and the normal equations follow from differentiation with respect to x, y, and z, as before :

$$\Sigma w_x a\,(w_x ax + w_y by + w_z cz + d) = 0,$$
$$\Sigma w_y b\,(w_x ax + w_y by + w_z cz + d) = 0,$$
$$\Sigma w_z c\,(w_x ax + w_y by + w_z cz + d) = 0,$$

being the fundamental error equation multiplied by the respective weight and coefficient of x, y, and z.

Solution may be (a) *direct* or (b) by *differences*. the procedure in the former method being as follows :

(1) Multiply each side of the observation equation by the weight of the observation, and form the normal equation of each quantity by multiplying the weighted observations containing that quantity by its algebraical coefficient. (2) Write down the normal equations for the several required quantities, and determine their individual and most probable values by simultaneous solution.

Example. Find the most probable values of the angles A, B, and C from the following observations at one station :

$$\begin{aligned} A &= 38^\circ\,12'\,26{\cdot}5'' \text{ ; weight } 1 \\ B &= 32^\circ\,45'\,13{\cdot}2'' \text{ ; } \quad ,, \quad 2 \\ A+B &= 70^\circ\,57'\,38{\cdot}6'' \text{ ; } \quad ,, \quad 2 \\ A+B+C &= 126^\circ\,28'\,00{\cdot}6'' \text{ ; } \quad ,, \quad 3 \\ B+C &= 88^\circ\,15'\,37{\cdot}8'' \text{ ; } \quad ,, \quad 1. \end{aligned}$$

(a) *Direct method.* Multiplying these observation equations by their

$$\begin{aligned} A &= 38^\circ\,12'\,26{\cdot}5'' \\ 2B &= 65^\circ\,30'\,26{\cdot}4'' \\ 2A+2B &= 141^\circ\,55'\,17{\cdot}2'' \\ 3A+3B+3C &= 379^\circ\,24'\,01{\cdot}8'' \\ B+\ C &= 88^\circ\,15'\,37{\cdot}8''. \end{aligned}$$

Forming the normal equations, first in A, by multiplying both sides by the algebraic coefficients of A in each :

$$\begin{aligned} A \qquad\qquad &= 38^\circ\,12'\,26{\cdot}5'' \\ 4A+\ 4B \qquad &= 283^\circ\,50'\,34{\cdot}4'' \\ 9A+\ 9B+9C &= 1138^\circ\,12'\,05{\cdot}4'' \\ \hline 14A+13B+9C &= 1460^\circ\,15'\,06{\cdot}3''. \qquad (1) \end{aligned}$$

Likewise the normal equation in B :

$$\begin{aligned} 4B \qquad\qquad &= 131^\circ\,00'\,52{\cdot}8'' \\ 4A+\ 4B \qquad &= 283^\circ\,50'\,34{\cdot}4'' \\ 9A+\ 9B+\ 9C &= 1138^\circ\,12'\,05{\cdot}4'' \\ B+\ C &= 88^\circ\,15'\,37{\cdot}8'' \\ \hline +18B+10C &= 1641^\circ\,19'\,10{\cdot}4''. \qquad (2) \end{aligned}$$

Finally, the normal equation in C:

$$\begin{aligned} 9A + \ \ 9B + \ \ 9C &= 1138^\circ\ 12'\ 05{\cdot}4'' \\ B + \ \ C &= \ \ 88^\circ\ 15'\ 37{\cdot}8'' \\ \hline 9A + 10B + 10C &= 1226^\circ\ 27'\ 43{\cdot}2' \end{aligned} \qquad \text{.(3)}$$

Solving these three simultaneous equations:
From (2) and (3):

$$\begin{aligned} 13A + 18B + 10C &= 1641^\circ\ 19'\ 10{\cdot}4'' \\ 9A + 10B + 10C &= 1226^\circ\ 27'\ 43{\cdot}2'' \\ \hline 4A + \ \ 8B \qquad &= \ \ 414^\circ\ 51'\ 27{\cdot}2''. \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

From (1) and (2):

$$\begin{aligned} 140A + 130B + 90C &= 14602^\circ\ 31'\ 03{\cdot}0'' \\ 117A + 162B + 90C &= 14771^\circ\ 52'\ 33{\cdot}6'' \\ \hline -\ 23A + \ \ 32B \qquad & \qquad 169^\circ\ 21'\ 30{\cdot}6'' \end{aligned}$$

From (a)

$$\begin{aligned} 32B \qquad &= \ 1659^\circ\ 25'\ 48{\cdot}8'' \\ \hline +\ 39A \qquad &= \ 1490^\circ\ 04'\ 18{\cdot}2'' \end{aligned}$$

Whence $A = 38^\circ\ 12'\ 25{\cdot}08''$.

From (a)

$$\begin{aligned} & \ \ 103^\circ\ 42'\ 51{\cdot}80'' \\ = & \ \ \ \ 38^\circ\ 12'\ 25{\cdot}08'' \\ 2B = & \ \ \ \ 65^\circ\ 30'\ 26{\cdot}72'' \end{aligned} \qquad B = 32^\circ\ 45'\ 13{\cdot}36''.$$

$$\begin{aligned} 9A + 10B + 10C &= 1226^\circ\ 27'\ 43{\cdot}20'' \\ 9A + 10B \qquad &= \ \ 671^\circ\ 23'\ 59{\cdot}32'' \\ \hline 10C &= \ \ 555^\circ\ 03'\ 43{\cdot}88''; \end{aligned} \qquad C = 55^\circ\ 30'\ 22{\cdot}39''.$$

(*b*) *By differences*. The procedure is less laborious if a set of values is assumed for the most probable values of the unknown quantities and the most probable series of errors are determined by normal equations, the errors thus found being added algebraically to the respective assumed values for the most probable values of the measurements.

Thus: (1) Assume the most probable values, taking single observations as they stand, and subtracting in the two or three variable equations for the others. Let d_A, d_B, etc., represent the unknown residual errors in the assumption. (2) Replace the observation equations by equations in terms of d_A, d_B, etc., to express the discrepancies between the observed results and those given by the assumed values, *always subtracting* the latter from the former. (3) Multiply the residual equations by the weights as before, and form the normal equations, multiplying each side by the coefficients of d_A, d_B, etc., in each of the weighted expressions. (4) Solve the simultaneous equations for the residuals, d_A, d_B, etc., and add these algebraically to the assumed values of the quantities.

The foregoing method is thus applied to the preceding example

$$
\begin{array}{rll}
\text{Assuming } A = & 38^\circ\ 12'\ 26{\cdot}5'' & \\
B = & 32^\circ\ 45'\ 13{\cdot}2'' & A + B = 70^\circ\ 57'\ 39{\cdot}7'' \\
C = & \underline{55^\circ\ 30'\ 24{\cdot}6''} & B + C = 88^\circ\ 15'\ 37{\cdot}8'' \text{ from obs.} \\
A + B + C = & 126^\circ\ 28'\ 04{\cdot}3'' &
\end{array}
$$

Reduced observation equations		Weighted equations
$d_A = 0{\cdot}0''$	; $w = 1.$	$d_A = 0{\cdot}0''.$
$d_B = 0{\cdot}0''$	; ,, 2.	$2d_B = 0{\cdot}0''.$
$d_A + d_B = -1{\cdot}1''$	; ,, 2.	$2d_A + 2d_B = -2{\cdot}2''.$
$d_A + d_B + d_C = -3{\cdot}7''$	; ,, 3.	$3d_A + 3d_B + 3d_C = -11{\cdot}1''.$
$d_B + d_C = 0{\cdot}0''$	; ,, 1.	$d_C = 0{\cdot}0''.$

Whence the normal equations

$$
\begin{array}{lrl}
\text{In } A, & 14d_A + 13d_B + 9d_C = -37{\cdot}7. & \ldots\ldots(1) \\
\text{In } B, & 13d_A + 18d_B + 10d_C = -37{\cdot}7. & \ldots\ldots(2) \\
\text{In } C, & 9d_A + 10d_B + 10d_C = -33{\cdot}3. & \ldots\ldots(3)
\end{array}
$$

$$
\begin{array}{lll}
\text{From (2) and (3):} & 4d_A + 8d_B & = -\ 4{\cdot}4 \quad \ldots\ldots(a) \\
\text{,, (1) ,, (2)} & 140d_A + 130d_B + 90d & = -337{\cdot}0 \\
 & \underline{117d_A + 162d_B + 90d} & = \underline{-339{\cdot}3} \\
 & +\ 23d_A - 32d_B & = -\ 37{\cdot}7 \\
 & \underline{16d_A + 32d_B} & = -\ \underline{17{\cdot}6} \\
 & 39d_A & = -\ 55{\cdot}3''; \quad d_A = -1{\cdot}42'' \\
\text{From } (a): & d_A + 2d_B & = -\ 1{\cdot}10''; \quad d_B = +0{\cdot}16'' \\
\text{From (3):} & 9d_A + 10d_B + 10d_C & = -33{\cdot}30 \\
 & \underline{9d_A + 10d_B} & = -11{\cdot}18 \\
 & C & = -22{\cdot}12 \ ; \ d_C
\end{array}
$$

Applying these to the assumed values:

$$
\begin{aligned}
A &= 38^\circ\ 12'\ 25{\cdot}08''. \\
B &= 32^\circ\ 45'\ 13{\cdot}36''. \\
C &= 55^\circ\ 30'\ 22{\cdot}39''.
\end{aligned}
$$

Correlates. When there is any fixed relationship between the variables to be satisfied, the inherent **equation of condition** has the effect of eliminating one of the variables. Likewise with m equations of condition, m of the variables must first be eliminated by expressing these in terms of the other variables by means of the m equations.

Suppose, for example, there is a level circuit with a closing error E, the weights by which the differences of the levels of the several bench-

marks are determined being w_1, w_2, w_3, w_4, etc. ; then if e_1, e_2, e_3, e_4, etc., are the necessary corrections to the observed level differences, there is one equation of condition ; namely,

$$\Sigma(e) = e_1 + e_2 + e_3 + e_4 + \ldots = \pm E. \quad \ldots\ldots\ldots\ldots(1)$$

Further, the least square condition requires that

$$\Sigma(we^2) = w_1e_1^2 + w_2e_2^2 + w_3e_3^2 + w_4e_4^2 + \ldots = \text{a minimum.} \quad \ldots\ldots\ldots(2)$$

Now if e_1, e_2, e_3, e_4, etc., be varied by increments δe_1, δe_2, δe_3, δe_4, etc.,

$$\Sigma(\delta e) = \delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 + \ldots = 0. \quad \ldots\ldots\ldots\ldots(3)$$

Then upon differentiating (2),

$$\Sigma(we \,.\, \delta e) = w_1e_1 \,.\, \delta e_1 + w_2e_2 \,.\, \delta e_2 + w_3e_3 \,.\, \delta e_3 + w_4e_4 \,.\, \delta e_4 + \ldots = 0. \quad \ldots\ldots(4)$$

Also multiplying (3) by $-\zeta$ and adding to (4),

$$(w_1e_1 - \zeta)\delta e_1 + (w_2e_2 - \zeta)\delta e_2 + (w_3e_3 - \zeta)\delta e_4 + (w_4e_4 - \zeta) \,.\, \delta e_4 + \ldots = 0. \ \ldots(5)$$

Since δe_1, δe_2, δe_3, δe_4, etc., are independent quantities, their coefficients must vanish independently, or

$$\zeta = w_1e_1 = w_2e_2 = w_3e_3 = w_4e_4 = \ldots ;$$

or

$$e_1, \frac{\zeta}{w_1} ; \ e_2 = \frac{\zeta}{w_2} ; \ e_3 = \frac{\zeta}{w_3} ; \ e_4 = \frac{\zeta}{w_4} ; \ e_5 = \ldots\ldots\ldots\ldots(6)$$

Hence, on substituting for e_1, e_2, e_3, e_4 ... in (1),

$$\zeta\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} + \ldots\right) = \pm E. \quad \ldots\ldots\ldots\ldots(7a)$$

If an additional line of levels is run from the initial benchmark to the fourth with an error e_0 and weight w_0, a closing error E_0 will be obtained from the original levels and the short circuit, so that

$$e_0 - (e_1 + e_2 + e_3) = \pm E_0. \quad \ldots\ldots\ldots\ldots(1a)$$

Thus (2) is replaced by a pair of equations :

$$\left.\begin{aligned} \Sigma(we^2) &= w_1e_1^2 + w_2e_2^2 + w_3e_3^2 + w_4e_4^2 + \ldots = \text{a minimum} \\ \Sigma(we^2)_0 &= w_0e_0^2 + w_1e_1^2 + w_2e_2^2 + w_3e_3^2 + \ldots = \text{a minimum} \end{aligned}\right\} \ldots\ldots\ldots(2a)$$

Also

$$\left.\begin{aligned} \delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 + \ldots &= 0 \\ \delta e_0 + \delta e_1 + \delta e_2 + \delta e_3 + \ldots &= 0 \end{aligned}\right\}. \ \ldots\ldots\ldots\ldots(3a)$$

And

$$\left.\begin{aligned} \Sigma(we \,.\, \delta e) &= w_1e_1 \,.\, \delta e_1 + w_2e_2 \,.\, \delta e_2 + w_3e_3 \,.\, \delta e_3 + w_4e_4 \,.\, \delta e_4 = 0 \\ \Sigma(we \,.\, \delta e)_0 &= w_0e_0 \,.\, \delta e_0 + w_1e_1 \,.\, \delta e_1 + w_2e_2 \,.\, \delta e_2 + w_3e_3 \,.\, \delta e_3 = 0 \end{aligned}\right\}. \quad (4a)$$

Multiplying (3a) by $-\zeta$ and $-\eta$ respectively and adding algebraically to (4a) :

$$\left.\begin{aligned} (w_1e_1 - \zeta)\delta e_1 + (w_2e_2 - \zeta)\delta e_2 + (w_3e_3 - \zeta)\delta e_3 + (w_4e_4 - \zeta)\delta e_4 + \ldots &= 0 \\ (w_0e_0 - \eta)\delta e_0 + (w_1e_1 - \eta)\delta e_1 + (w_2e_2 - \eta)\delta e_2 + (w_3e_3 - \eta)\delta e_3 + \ldots &= 0 \end{aligned}\right\}. \quad (5a)$$

Adding the last equations algebraically :

$$(2w_1e_1 - \zeta - \eta)\,\delta e_1 + (2w_2e_2 - \zeta - \eta)\,\delta e_2 + (2w_3e_3 - \zeta - \eta)\,\delta e_3$$
$$(w_4e_4 - \zeta)\,\delta e_4 + (w_0e_0 - \eta)\,\delta e_0 + \ldots\ldots = 0.$$

For the coefficients of δe_0, δe_1, ... , to vanish independently ;

$$e_0 = \frac{\eta}{w_0}\,; \quad e_2 = \frac{\zeta+\eta}{2w_2}\,; \quad e_3 = \frac{\zeta+\eta}{2w_3}\,; \quad e_4 = \frac{\zeta}{w_4}\,; \; e_5 = \ldots\ \ldots \quad \ldots(6a)$$

Finally, on substituting in (1) and (1a) :

$$\left.\begin{aligned} &\frac{\zeta+\eta}{2w_1} + \frac{\zeta+\eta}{2w_2} + \frac{\zeta+\eta}{2w_3} + \frac{\zeta}{w_4} + \ldots = E \\ &\frac{\eta}{w_0} - \frac{\zeta+\eta}{2w_1} - \frac{\zeta+\eta}{2w_2} - \frac{\zeta+\eta}{2w_3} + \ldots \quad - \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7a)$$

In many problems, however, the process is simplified by the use of the simple correlate expressions of (7) with appropriate weighting of combinations of level differences, or angles, as the case may be.

Weight. As already stated, the weight is the inverse square of the probable error, and may be assessed *empirically*, or fixed theoretically by the number n of the measurements of the quantity. Thus, if e is the error in a single measurement, the error in the mean of n equally accurate measurements will be $e/\sqrt{n}$, and its weight will be n, giving the error square term as $n \,.\, e^2$. Conversely, if a quantity is the sum of n equally accurate measurements, the weight will be $1/n$ in the error square term, e^2/n. In levelling operations, the weight may be determined by the number of times the section or circuit is run, or the staff readings are repeated, and inversely as the number of settings up of the instrument, which is explicit in the distance, since balanced back and foresights of approximately uniform length would characterise work in which the present considerations would be involved.

Example†. The following round of angles was observed from a central station to the surrounding stations of a triangulation survey :

$\theta = 93°\ 43'\ 20''$; $\phi = 74°\ 32'\ 40''$; $\psi = 101°\ 13'\ 45''$; $\omega = 90°\ 29'\ 55''$.

In addition, one angle $(\theta + \phi)$ was measured separately twice as a combined angle with a mean value of $168°\ 16'\ 05''$.

Determine the most probable values of the angles θ, ϕ, ψ, and ω, assuming that all the *six* measurements are equally precise.

As in the case of many problems, this may be solved by normal equations by eliminating ω, writing $\theta + \phi + \psi = 360° - \omega = 269°\ 30'\ 05''$, leading thus to the following normal equations in θ, ϕ, and ψ :

$$4\theta + 3\phi + \psi = 699°\ 45'\ 35''\ ; \quad 3\theta + 4\phi + \psi = 680°\ 34'\ 55''\ ;$$
$$\theta + \phi + 2\psi = 370°\ 43'\ 50''.$$

Such a proceeding is usually tedious, and the solution by correlates will be given (*a*) from first principles, and (*b*) by formulae, the latter introducing quick methods of solution.

(*a*) *From first principles.* $\theta+\phi+\psi+\omega=360^\circ$; correction $+20''$; $\overline{(\theta+\phi)}=\theta+\phi$; correction, $-5''$. Let $e_1, e_2, e_3, e_4,$ [illegible]$_5$ be the corrections in order. Then $e_1^2+e_2^2+e_3^2+e_4^2+2e_5^2=$ a minimum.

(1) $e_1+e_2+e_3+e_4=+20''$; (2) $e_5-e_1-e_2=$ [illegible].

$$e_1\,.\,\delta e_1+e_2\,.\,\delta e_2+e_3\,.\,\delta e_3+e_4\,.\,\delta e_4+2e_5\,.\,\delta e_5=0,$$

$$\delta e_1+\delta e_2+\delta e_3+\delta e_4=0. \quad \text{Mult. by } -\zeta.$$

$$-\delta e_1-\delta e_2+\delta e_5 \qquad =0. \quad \text{,,} \quad \text{,,} \quad -\eta.$$

On equating the coefficients to zero, $e_1=\zeta-\eta=e_2$; $e_3=\zeta=e_4$; $e_5=\frac{1}{2}\eta$.

Substituting in (1), $4\zeta-\ \ 2\eta=+20''.$ (2), $-2\zeta+2{\cdot}5\eta=-5''.$ Whence $\eta=3\frac{1}{3}$, $\zeta=6\frac{2}{3}$,

and $e_1=e_2=3\frac{1}{3}''$; $e_3=e_4=6\frac{2}{3}''$; $e_5=1\frac{2}{3}''$,

leading to

$\theta=93^\circ\,43'\,23\frac{1}{3}''$; $\phi=74^\circ\,32'\,43\frac{1}{3}''$; $\psi=101^\circ\,13'\,51\frac{2}{3}''$; $\omega=90^\circ\,30'\,01\frac{2}{3}''$,

with combined angle $\overline{(\theta+\phi)}=168^\circ\,16'\,06\frac{2}{3}''$.

(*b*) *By formulae.* (i) Combine $(\theta+\phi)$ as

$$(93^\circ\,43'\,20''+74^\circ\,32'\,40'')=168^\circ\,16'\,00''$$

in the first series with weight w_1, which will be $\frac{1}{2}$ since the p.e. in the sum of the angles will be $\sqrt{2}e$, where e is the *common* angle error of $3''$ in this round. Now if this be combined with $\overline{(\theta+\phi)}$ as observed with weight $w_2=2$, the resultant weight w will be $2\frac{1}{2}$ with the following value of $(\theta+\phi)$:

$$\frac{w_1(\theta+\phi)+w_2\overline{(\theta+\phi)}}{w_1+w_2}=168^\circ\,16'\,04'' \text{ weight } 2\tfrac{1}{2}$$

(ii) $\psi=101^\circ\,13'\,45''$,, 1

(iii) $\omega=\ \ 90^\circ\,29'\,55''$,, 1

$359^\circ\,57'\,44''$; $e=-16''$.

Now $e_3=e_4=\dfrac{1}{\frac{1}{\frac{1}{2}}+1+1}\cdot 16''$

$=6\frac{2}{3}$, added to given values of $\psi+\omega$, while $e_{(1+2)}=+2\frac{2}{3}''$.

But $2''$ is to be added to each of the values of θ and ϕ to make the compound angle in (i) with $w=2\frac{1}{2}$, while a further addition of $1\frac{1}{3}''$ follows from the correlates. Hence the foregoing final values of θ, ϕ, ψ, and ω.

Example†. In establishing a system of benchmarks for a main drainage scheme, the levels were based upon an Ordnance B.M.—*O*—at the Municipal Buildings, and the reduced levels of three primary benches

P, Q, and R were determined by precise levelling as the basis of the system.

The first three values are direct differences of elevation above O, and the remaining ones are independent differences between the outer benchmarks.

$$
\begin{aligned}
P &= +28{\cdot}018 \text{ with weight } 5\\
Q &= +16{\cdot}542 \quad ,, \quad ,, \quad 5\\
R &= -14{\cdot}280 \quad ,, \quad ,, \quad 2\\
P-Q &= +11{\cdot}454 \quad ,, \quad ,, \quad 3\\
Q-R &= +30{\cdot}810 \quad ,, \quad ,, \quad 4
\end{aligned}
$$

Given that the reduced level of O is 64·60, determine the most probable values of the benchmarks P, Q and R. (U.L.)

Here the normal equations are as follows :

$$
\begin{aligned}
&\text{In } P, \quad 34P-9Q &&= +\ 803{\cdot}536. \ldots\ldots\ldots\ldots(1)\\
&\text{In } Q, \quad 9P+32Q-16R &&= +1009{\cdot}596. \quad \ldots\ldots\ldots(2)\\
&\text{In } R, \quad 16Q-12R &&= +\ 435{\cdot}840. \ldots\ldots\ldots\ldots(3)
\end{aligned}
$$

Simultaneous solution leads to the following values: $P=28{\cdot}013$; $Q=16{\cdot}560$; $R=-14{\cdot}250$, with the following reduced levels :—

$$P=92{\cdot}613\ ;\ Q=81{\cdot}160\ ;\ R=50{,}350.$$

Example†. In observing a round of angles at a central station with a repeating theodolite, the sets were measured partly as individual angles and partly as single composite angles, the latter being taken from the referring point of the first angle α.

Determine the most probable values of the angles without resorting to the use of normal equations.

Angle	Mean value	Weight
α	68° 16′ 42″	5
β	72° 36′ 25″	3
γ	94° 16′ 19″	3
δ	124° 50′ 38″	3
$\alpha+\beta$	140° 53′ 07″	2
$\alpha+\beta+\gamma$	235° 09′ 25″	2
$\alpha+\beta+\gamma+\delta$	360° 00′ 03″	2

(U.L.)

Separate into two series and use α common to both if necessary.

(1) Merely adjust the individual angles in the first : $+4''$ error.

$$e_\alpha=\frac{\frac{1}{5}}{\frac{1}{5}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}4=+0{\cdot}67''\ ;\ e_\beta=e_\gamma=e_\delta=+1{\cdot}11''.$$

Tabulate the corrected angles :

$$16'\ 41{\cdot}33'',\ 36'\ 23{\cdot}89'',\ 16'\ 17{\cdot}89'',\ 50'\ 36{\cdot}89''.$$

(2) In the second series all the angles have the same weight, and therefore a common probable error $\pm e$. This is $+3''$ for $\alpha+\beta+\gamma+\delta$, but, since

either the plus or minus sign may obtain, the other composite values are the most probable. Also the most probable value of α is its corrected value in the first series, since otherwise, with the observed value of α, the final values would not sum up exactly. Hence subtract this value from $\alpha+\beta$ for β; and subtract the remaining composite values for γ and δ, using 360° in the latter case.

Tabulate the corrected angles, omitting the degrees, as before :

$$16'\,41{\cdot}33''\,; \quad 36'\,25{\cdot}67''\,; \quad 16'\,18''\,; \quad 50'\,35''.$$

Introduce the corrected values in both series in $\dfrac{w_1A_1+w_2A_2}{w_1+w}$, the corrected value of α in the first series obtaining while β is 72° 36′ 25·67″.

Tabulate :

$$\alpha=41{\cdot}330''\,;\ \beta=24{\cdot}602''\,;\ \gamma=17{\cdot}934''\,;\ \delta=36{\cdot}134''\,:\ \text{total, } 120{\cdot}0''.$$

Example††. The following notes refer to the precise levelling of a city benchmark system, being the outer circuit closing upon B.M. 1.

In addition to the mean observed reduced levels and the distances M in miles between the benchmarks, the numbers N of settings-up in the several sections are given together with the numbers n of times those sections were run, balanced backsights and foresights being used in all cases.

(a) Determine the most probable reduced levels of the B.M.'s, appropriately weighting the observations and using N in preference to M.

(b) Determine the values of the coefficient c in the several sections when the probable errors are expressed as

B.M.	Mean obs. R.L.	M (miles)	N	n
1	172·600			
		1·20	12	4
2	182·822			
		0·44	4	2
3	204·446			
		1·72	16	4
4	212·662			
		2·22	24	8
5	194·344			
		0·64	6	2
6	186·482			
		0·96	8	4
7	180·426			
		0·78	8	2
1	172·635			

(U.L.)

B.M.	Weight w	Correction	Coeff. c	Corr. R.L.
1		$-0{\cdot}0050$		172·600
	$\frac{1}{3}$		0·0045	
2		$-0{\cdot}0033$		182·817
	$\frac{1}{2}$		0·0051	
3		$-0{\cdot}0066$		204·438
	$\frac{1}{4}$		0·0050	
4		$-0{\cdot}0050$		212·647
	$\frac{1}{3}$		0·0036	
5		$-0{\cdot}0050$		194·324
	$\frac{1}{3}$		0·0063	
6		$-0{\cdot}0033$		186·457
	$\frac{1}{2}$		0·0034	
7		$-0{\cdot}0066$		180·398
	$\frac{1}{4}$		0·0075	
1		$-0{\cdot}0348$ ft.		172·601

Solution by $\zeta\left(\frac{1}{w_1}+\frac{1}{w_2}+\frac{1}{w_3}+\ldots\right)=\pm E=-0{\cdot}035$;

$$e_1=\frac{1/w_1}{\Sigma(1/w)}E, \text{ etc.}$$

Corrections are added algebraically to the rises (+) and the falls (−) between the observed reduced levels; and the differences of R.L. thus corrected are applied to the preceding corrected R.L.'s.

Example†. The following data were obtained in observations with an internal-focusing telescope, which was stated to be anallatic with a multiplier of 100, the telescope being level during the tests:

Hor. dist. -	80	120	160	200	240	280	320	360 ft.
Intercept -	0·79	1·20	1·62	1·99	2·42	2·81	3·18	3·59 ft.

Determine the most probable value of the multiplier and the additive correction, if any. (U.L.)

Assuming the relation $D=ks+C$, where s is the intercept, k the multiplier, and C the anallatic correction; then the error

$$\Sigma\{(80-0{\cdot}79k-C)+(120-1{\cdot}2k-C)+\ldots\}^2 \text{ is to be a minimum;}$$

and on differentiating first with respect to k,

$$\begin{aligned}&0{\cdot}79(80-0{\cdot}79k-C)+1{\cdot}2(120-1{\cdot}2k-C)+1{\cdot}62(160-1{\cdot}62k-C)\\&\quad+1{\cdot}99(200-1{\cdot}99k-C)+2{\cdot}42(240-2{\cdot}42k-C)\\&\quad+2{\cdot}81(2{\cdot}80-2{\cdot}81k-C)+3{\cdot}18(320-3{\cdot}18k-C)\\&\quad+3{\cdot}59(360-3{\cdot}59k-C)=0,\end{aligned}$$

giving $$4542-45{\cdot}40k-1760C=0. \qquad \ldots\ldots\ldots\ldots(1)$$

Now differentiating with respect to C,

$$1(80-0{\cdot}79k-C)+1(120-1{\cdot}2k-C)+1(160-1{\cdot}62k-C)+\ldots=0,$$

giving $$1760-17{\cdot}60k-8C=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

From (1) and (2) : $C=\dfrac{2782-27{\cdot}8k}{9{\cdot}6}=289{\cdot}782-2{\cdot}896k. \quad \ldots\ldots \quad .(3)$

Substituting in (1) :

$$4542=45{\cdot}4k+17{\cdot}6(289{\cdot}782-2{\cdot}896k),$$

with $$k=100{\cdot}208.$$

Substituting for k in (3), $C=289{\cdot}782-290{\cdot}202=-0{\cdot}42$ ft.

QUESTIONS ON ARTICLE 2

1†. The following are the mean values observed in the measurement of three angles α, β and γ at one station :

α	76° 42′ 46·2″	with weight	4
$\alpha+\beta$	134° 36′ 32·6″	,,	3
	185° 35′ 24·8″	,,	2
	262° 18′ 10·4″	,,	1

Calculate the most probable value of each angle. (U.L.)

[$\alpha=76°\ 42'\ 46{\cdot}17''$; $\beta=57°\ 53'\ 46{\cdot}43''$; $\gamma=127°\ 41'\ 38{\cdot}26''$.]

2†. The following observations of three angles, A, B, and C were taken at one station :

$A=$	75° 32′ 46·3″	with weight	3
$B=$	55° 09′ 53·2″	,,	2
$A+B=$	130° 42′ 41·6″	,,	2
$B+C=$	163° 19′ 22·5″	,,	1
$A+B+C=$	238° 52′ 9·8″	,,	1

Determine the most probable value of each angle. (U.L.)

[$A=75°\ 32'\ 46{\cdot}71''$; $B=55°\ 09'\ 54{\cdot}05''$; $C=108°\ 09'\ 28{\cdot}75''$]

3†. A base line AE, approximately 2 miles in length, was measured in segments by means of the following apparatus, a secondary object being that of investigating the relative degrees of accuracy of the methods :

Segment	Apparatus	Observed length (ft.)	Estimated probable error
AB	Colby's Compensated Bars	2503·3760	1 in $1{\cdot}5\times10^6$
BC	Eimbeck Duplex Bars	2132·4340	1 ,, 0·8 ,,
AC	Guillaume-Carpentier Wires	4636·1540	1 ,, 1·0 ,,
AD	Invar Tape on Tripods	7246·5080	1 ,, 0·6 ,,
CE	U.S.A. C. & G. Apparatus	5918·4320	1 ,, 1·2 ,,

Determine the most probable length of the base line. (U.L.)

[10,554·9995 ft.]

4†. The following results were recorded in running a circuit of precise levels for four primary B.M.'s from an initial benchmark I, the weights of the levellings between successive B.M.'s being as indicated. Determine the probable reduced levels of II, III and IV.

B.M.	Observed reduced level	Weights	Corrected reduced level
I	100·000		
II	105·822	2	(105·813)
III	110·526	1	(110·499)
IV	113·118	2	(113·082)
I	100·054	1	

[Corrected values shown in last column.] (U.L.)

5†. A and C are two benchmarks, station C being 86·95 ft. above A. Precision levelling was carried out to find the level of two stations B and D and runs of levels were taken between the stations as follows :

End stations	Length	Rise	Fall
$A-B$	3 miles	37·60	
$B-C$	4 miles	49·64	
$C-D$	4 miles		30·64
$D-B$	6 miles		18·71
$D-A$	3 miles		56·14

Assuming the weights of the observed differences of level are inversely proportional to the lengths of the lines, compute the most probable levels of B and D. (U.L.)

[B : 37·497 ; D : 56·201 ft. above A]

6†. Derive from the principles of Least Squares an expression for the correction of triangles when the weights of the observations of the angles are unequal.

The following values were recorded for a triangle ABC, the individual measurements being uniformly precise :

$A = 72° \ 12' \ 18''$; 8 obs.
$B = 46° \ 40' \ 34''$; 6 obs.
$C = 61° \ 06' \ 56''$; 4 obs.

Give the corrected values of the angles. (U.L.)

[$A = 72° \ 12' \ 20·8''$; $B = 46° \ 40' \ 37·7''$; $C = 61° \ 07' \ 01·5''$]

7†. The following are the mean values of the four angles comprising the round of angles at a central station, the varying weights following from the method by which the angles are repeated :

α	76° 42′ 46·2″	with weight	8
$\alpha+\beta$	134° 36′ 32·6″	,,	7
	262° 18′ 10·4″	,,	6
$\gamma+o$	360° 00′ 08·8″	,,	5

Determine the most probable value of each angle. (U.L.)

[The most probable value of $\alpha+\beta+\delta+\gamma$ is 360°, and, although the errors in the observed values can be expressed, the most probable values are as stated and the values of the individual angles are found by subtraction, the last value being put at 360°. This fact may be verified by writing down the normal equations for the first three angles ; namely,

$$149\alpha+85\beta+36\gamma=20948°\ 22'\ 08{\cdot}6'' ;$$
$$85\alpha+85\beta+36\gamma=16038°\ 44'\ 51{\cdot}8'' ;$$

and

$$36\alpha+36\beta+36\gamma=\ 9442°\ 54'\ 14{\cdot}4'' ;$$

and solving in the usual way.]

8. The angles of a geodetic triangle were read as follows :

$A=50°\ 23'\ 17{\cdot}12''$ average of 6 observations.

$B=64°\ 24'\ 30{\cdot}16''$ average of 8 observations.

$C=65°\ 12'\ 15{\cdot}22''$ average of 10 observations.

If the area of this triangle is 755 square miles, adjust the angles A, B and C in the usual manner.

Adjust the angles x and y, readings of which give

$$x=25°\ 15'\ 25'', \text{ weight } 4.$$
$$y=32°\ 17'\ 41'', \text{ weight } 2.$$
$$x+y=57°\ 33'\ 12'', \text{ weight } 6.$$

What is Legendre's Theorem?

Give in their correct sequence, the various steps in calculating the two unknown sides of a geodetic triangle of which the three angles have been read, each being weighted differently and of which the length of one side is known. (U.B.)

[Errors all minus since excess $= +9{\cdot}95''$: $3{\cdot}17''$; $2{\cdot}38''$; $1{\cdot}90''$;
$50°\ 23'\ 20{\cdot}29''$; $64°\ 24'\ 32{\cdot}54''$; $65°\ 12'\ 17{\cdot}12''$.]

9. (*a*) Explain briefly the principle of the prismatic astrolabe and describe the method for which it has been primarily designed, of obtaining the latitude of a place by equal altitudes of two stars.

(*b*) Adjust the angles a and b, observations of which give :

$$a=20°\ 10'\ 10'', \text{ weight } 6.$$
$$b=30°\ 20'\ 30'', \text{ weight } 4.$$
$$a+b=50°\ 30'\ 50'', \text{ weight } 2.$$

(*c*) Explain " spherical excess " and give a simple equation for calculating its value in seconds for a triangle of N square miles. (U.B.)

[(*b*) $a=20°\ 10'\ 02''$, (*c*) $\log.\ \epsilon''=\log.\ N+\bar{2}{\cdot}119858$.]
$b=30°\ 20'\ 48''$;

ARTICLE 3: ADJUSTMENT OF TRIANGLES

The adjustment of the angles of a triangulation net may be said to be (*a*) simple, (*b*) compound, and (*c*) rigid.

(*a*) **Simple chain treatment** precludes the log sine or side condition and is sometimes applied over areas of considerable extent, even when spherical triangles are involved. It possesses the advantage of simplicity, allowing appropriate weighting of angular observations without difficulty.

(*b*) **Compound methods** combine the geometrical conditions with the log sine condition, and the process is readily applicable to systems comprised of quadrilaterals with intersecting diagonals and polygons with central stations. Adjustment on these lines is occasionally applied to nets of considerable size in order to avoid the enormous labour involved by the rigorous use of least squares. The artifice of **equal shifts** not only simplifies the computations, but also renders the method systematic, though unfortunately the method is devoid of mathematical justification, in so far as the theory of errors is concerned. Nevertheless, it is a definite means to an end, though actually it gives only one of an indefinite number of solutions in any given case. Many, however, prefer to proceed by trial and error, guided by the relative effects of the log sine differences (see Art. 4).

(*c*) **Rigid method.** The most accurate method is that of least squares, and the most rigid application follows when the entire system is adjusted in one mass, all the angles being simultaneously involved. In addition to the geometrical and log sine conditions, the sums of the squares of the errors must be a minimum. The process is exceedingly laborious, even in nets comprising few figures. In the Ordnance Survey there were 920 equations of condition, and if the whole had been treated as a single entity the same number of unknown quantities would have been involved as a part of the work. Avoiding this, the triangulation was divided into twenty-one figures, and four of these, not adjacent, were adjusted by another method, leading to corrections which were embodied in the equations of condition of the adjacent figures. In this way the equations were reduced to a maximum of 77, with an average of 44 in a figure. Despite the enormous amount of work involved, the method of least squares has been employed in notable surveys, including the great Survey of India (see Art. 4, p. 425).

The present article deals with the simple chain treatment, but the methods described may be subsidiary to adjustment by more rigorous methods.

Notation. The following symbols will be used in this connection, appropriate subscripts being applied, as indicated.

e, the total error in a triangle, with e_1, e_B, etc., the errors in the individual angles.

n_1, n_A, the number of observations of an angle.

A_1, B_2, the individual observation of an angle, with A_0, B_0, the mean of the observed values.

w_3, w_A, the weights of the observation of angles.

ϵ, the spherical excess, which is $\dfrac{S}{\pi R^2} \times 180° = \dfrac{S}{75{\cdot}5}$ approximately, with S in square miles ; or in seconds, $\log \epsilon'' = \log S' - 9{\cdot}3254098$, with S' the area in square feet, the earth's radius R being taken at 20,889,000 ft. Whence $e = 180° + \epsilon - (A_0 + B_0 + C_0)$.

Although various rules for correction are given, the rational method follows from correlates, as described on p. 408 :

$$e \propto \frac{1}{w_A} + \frac{1}{w_B} + \frac{1}{w_C}\,; \quad e_A \propto \frac{1}{w_A}\,; \quad e_A = \left(\frac{1/w_A}{1/w_A + 1/w_B + 1/w_C}\right) e.$$

This may be regarded fundamental for plane and spherical triangles.

(1) If all the observations are considered to be equally precise, that is, of equal weight, the above rule indicates that $\pm \frac{1}{3}e$ is to be applied to each angle.

(2) If the values A_0, B_0, C_0 are averages of unequal numbers of equally precise observations, the errors may be distributed in accordance with the following rule, which is based upon the fact that if an angle is measured n times with equal accuracy, the average value is equivalent to $\sqrt{n}$ and the error squared term is ne^2, n being the weight.

(*a*) *Inverse corrections.*

$$e \propto \frac{1}{n_A} + \frac{1}{n_B} + \frac{1}{n_C}\,; \quad e_A \propto \frac{1}{n_A}\,; \quad e_A = \left(\frac{1/n_A}{1/n_A + 1/n_B + 1/n_C}\right) e \text{ (see Note, p. 420).}$$

(*b*) *Inverse square corrections* are sometimes cited, but there appears to be little mathematical justification for the rule, which is as follows :

$$e_A = \left(\frac{1/n_A{}^2}{1/n_A{}^2 + 1/n_B{}^2 + 1/n_C{}^2}\right) e.$$

(3) If the mean values are not equally precise and preclude weighting by number, the fundamental rule must be applied ; thus :

$$e_A = \left(\frac{1/w_A}{1/w_A + 1/w_B + 1/w_C}\right) e.$$

Normally the weights are understood to be the inverse squares of the probable errors, E_A, E_B, etc., and the foregoing equation reduces to

$$e_A = \left\{\frac{E_A{}^2}{E_A{}^2 + E_B{}^2 + E_C{}^2}\right\} e \text{ (see Note, p. 420).}$$

Sometimes the weights are derived from the average error, $\pm\frac{\Sigma d}{n}$, or the mean square error, $\pm\sqrt{\frac{\Sigma d^2}{n}}$, where d is the residual, or difference between A_0 and A_1, A_2, etc.

Gauss's Rule. Here the weights are taken as $w_A = \frac{\frac{1}{2}n^2}{\Sigma d^2}$, etc., and are introduced in the general rule.

(*c*) *Inverse square weight corrections.* The following is a rule for which there is little mathematical justification :

$$e_A = \left\{\frac{(1/w_A)^2}{(1/w_A)^2 + (1/w_B)^2 + (1/w_C)^2}\right\} e.$$

Note. Although 2 (*a*) and 3 are identical in the abstract, they will not give identical results when the weights are mere numbers on one hand, and are derived from E_A, E_B, E_C for actual observations on the other. If the weights can be derived from the observations, these are to be preferred, but only in so far as estimates of probable errors are warranted by the numbers of observations involved (see p. 400).

Example†. The following angles of a spherical triangle *PQR*, 150 square miles in area were observed under the same conditions as regards accuracy of measurement :

P : 55° 48′ *plus* 20″, 27″, 22″, 26″, 24″, 26″, 27″, 20″.
Q : 62° 25′ *plus* 14″, 18″, 12″, 20″, 19″, 13″.
R : 61° 46′ *plus* 18″, 34″, 30″, 22″.

Adjust these angles on the assumption that the distribution of error should be in the ratio of the reciprocals of the weights of the observations, the latter values being determined by the theory of least squares.

N.B.—Spherical excess e'' may be determined by the following relation with the area s of the spherical triangle in square miles,

$$\log e'' = \log s + \bar{2}{\cdot}119850. \qquad \text{(U.L.)}$$

Strictly the wording of the question admits of two solutions, although only the mean angles and numbers of observations are required in the alternative case.

Here the rule $e_A = \left(\frac{1/w_A}{1/w_A + 1/w_B + 1/w_C}\right) e$ becomes $\left(\frac{E_A{}^2}{E_A{}^2 + E_B{}^2 + E_C{}^2}\right) e$, where e_A is the particular error and e is the total error in the spherical triangle.

P (8 obs.) with a mean angle of 55° 48′ 24″. Residuals $d = \pm 4, 3, 2$, etc.
$d^2 = 16\ 9\ 4\ 4\ 0\ 4\ 9\ 16$; $\Sigma d^2 = 62$.
Q (6 obs.) with a mean angle of 62° 25′ 16″. $d = \pm 2, 2, 4, 4, 3, 3$.
$d^2 = 4\ 4\ 16\ 16\ 9\ 9$;

R (4 obs.) with a mean angle of $61° 46' 26''$. $d = \pm 8, 8, 4, 4$.

$$d^2 = 64 \quad 64 \quad 16 \quad 16; \qquad \Sigma d^2 = 160.$$

$$E_P \propto \sqrt{\frac{62}{8\times 7}} = 1{\cdot}05; \quad E_Q \propto \sqrt{\frac{58}{6\times 5}} = 1{\cdot}39; \quad E_R \propto \sqrt{\frac{160}{4\times 3}} = 3{\cdot}65.$$

$$E_P{}^2 \propto 1{\cdot}10. \qquad E_Q{}^2 \propto 1{\cdot}93. \qquad E_R{}^2 \propto 13{\cdot}32.$$

For the spherical excess,

$$\log e'' = \log 150 + \bar{2}{\cdot}1198580 = 0{\cdot}2959493; \; e'' = 1{\cdot}977''; \text{ say } 2''.$$

Sum of the mean angles $P, Q, R = 180° \, 0' \, 6''$, leaving $4''$ (subtractive) for distribution by the method specified. $E_P{}^2 + E_Q{}^2 + E_R{}^2 = 16{\cdot}35$.

$$e_P = \frac{1{\cdot}10\times 4}{16{\cdot}35} = 0{\cdot}27''; \; e_Q = \frac{1{\cdot}93\times 4}{16{\cdot}35} = 0{\cdot}47''; \; e_R = \frac{13{\cdot}32\times 4}{16{\cdot}35} = 3{\cdot}26'',$$

giving a total of $4{\cdot}0'' = e$.

Whence the corrected mean values:

$$P = 58° \, 48' \, 23{\cdot}73''; \quad Q = 62° \, 25' \, 15{\cdot}53''; \quad R = 61° \, 46' \, 22{\cdot}74''.$$

$$\text{Total}: \quad 180° \, 00' \, 2{\cdot}0'' = 180° + e''.$$

An alternative solution might be obtained by avoiding the p.e. formula and assuming the relative weights to be the number of observations, thus:

$$e_P = \frac{(1/n_P)e}{1/n_P + 1/n_Q + 1/n_R} = \frac{0{\cdot}125\times 4}{0{\cdot}541} = 0{\cdot}92''; \quad e_Q = \frac{0{\cdot}166\times 4}{0{\cdot}541} = 1{\cdot}22'';$$

$$e_R = \frac{0{\cdot}25\times 4}{0{\cdot}541} = 1{\cdot}86''. \quad \text{Total}: \; e = 4{\cdot}0''.$$

QUESTIONS ON ARTICLE 3

1†. The following are the mean values of the angles A, B, and C of a plane triangle which were measured respectively eight, four, and six times:

$$54° \, 12' \, 25{\cdot}00''; \quad 48° \, 46' \, 16{\cdot}25''; \quad 77° \, 2' \, 10{\cdot}83''.$$

Find the corrected values of the angles by (*a*) least squares, and (*b*) Gauss's rule, using inverse weights in each case.

[(*a*) $54° \, 12' \, 12{\cdot}97''$; $48° \, 45' \, 52{\cdot}19''$; $77° \, 1' \, 54{\cdot}82''$
(*b*) $54° \, 12' \, 00{\cdot}64''$; $48° \, 45' \, 58{\cdot}93''$; $77° \, 2' \, 00{\cdot}41''$]

2†. The following are the observed values of the angles of a plane triangle:

$$A = 89° \, 49' \, 38''; \quad B = 60° \, 33' \, 26''; \quad C = 29° \, 37' \, 24''.$$

Adjust these to the nearest second on the assumption that the correction to any angle will be proportional to the sum of the reciprocals of the including sides.

[Here $\quad e = -28'' = e_A + e_B + e_C \propto 2(\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C) = 8{\cdot}342$;

$$e_A = \left(\frac{\operatorname{cosec} B + \operatorname{cosec} C}{\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C}\right) e = -\frac{3{\cdot}171}{8{\cdot}342} \times 28'' = -11'';$$

likewise $e_B = -10''$; $e_C = -7''$.]

ARTICLE 4 : ANGLES IN MINOR TRIANGULATION

There are *three* geometrical equations of conditions in the polygons comprising the triangles of a minor triangulation, which may be stated as follows with reference to Fig. 166.

(1) **Apex condition.** Sum of all the angles around a common vertex must be equal to four right angles.

$$a = 360° - (1 + 2 + 3 + 4) = e_1 + e_2 + e_3 + e_4.$$

(2) **Triangle condition.** Sum of the angles of individual triangles to be equal to two right angles.

$$b_1 = 180° - (1 + 5 + 6) = e_1 + e_5 + e_6\ ; \quad b_2 = e_4 + e_7 + e_8, \text{ etc.}$$

From these it follows that the sum of the interior angles will be equal to $(2n - 4)$ right angles, where n is the number of sides in the polygon.

Thus the polygon error

$$c = (e_5 + e_{12}) + (e_6 + e_7) + (e_8 + e_9) + (e_{10} + e_{11})$$

and $$a + c = b_1 + b_2 + b_3 + b_4 = \Sigma e.$$

(3) **Log sine condition.** The foregoing conditions may be fulfilled with the outer sides discontinuous but parallel to their correct positions, as indicated by the dotted lines. Hence the necessity of a third condition, which follows from the fact that the **sum of the log sines of the left-hand angles must equal the sum of the log sines of the right-hand angles**, these angles being so designated as they appear to the left or right of an observer who traverses the figure, always facing the common vertex or central station O.

Two cases will arise : (i) When the common vertex O is a central station, and (ii) when the common vertex O is an exterior station.

(i) Thus in Fig. 166, the even numbers defining the right-hand angles ; the odd, the left-hand angles.

$$\frac{OA}{OB} = \frac{\sin 6}{\sin 5}\ ; \quad \frac{OB}{OC} = \frac{\sin 8}{\sin 7}\ ; \quad \frac{OC}{OD} = \frac{\sin 10}{\sin 9}\ ; \quad \frac{}{OA} = \frac{}{\sin 11},$$

and on taking the product of the ratios of the sides,

$$\frac{OA}{OA} = 1 = \frac{\sin 6 \times \sin 8 \times \sin 10 \times \sin 12}{\sin 5 \times \sin 7 \times \sin\ 9 \times \sin 11} = \frac{\text{Product of sines of R.H. angles}}{\text{Product of sines of L.H. angles}}.$$

Whence :

$$\log \sin 6 + 8 \log \sin + \ldots = \log \sin 5 + \log \sin 7 + \ldots .$$

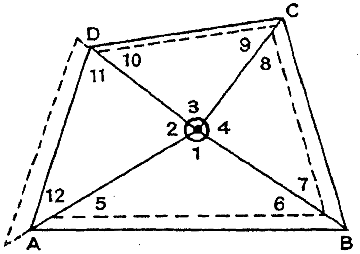


Fig. 166.

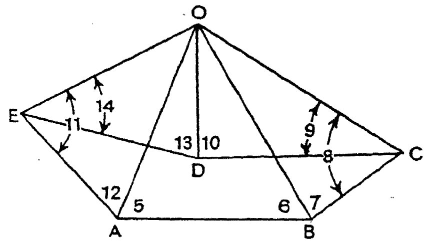


Fig. 167.

(ii) When the common vertex is outside, as in Fig. 167, the same rule applies :

$$\frac{OA}{OB}=\frac{\sin 6}{\sin 5}; \quad \frac{OB}{OC}=\frac{\sin 8}{\sin 7}; \quad \frac{OC}{OD}=\frac{\sin 10}{\sin 9}; \quad \frac{OD}{OE}=\frac{\sin 14}{\sin 13}; \quad \frac{OE}{OE}=\frac{\sin 12}{\sin 11};$$

and on taking the product of the ratios of the sides :

$$\frac{OA}{OA}=\frac{\sin 6\times\sin 8\times\sin 10\times\sin 12\times\sin 14}{\sin 5\times\sin 7\times\sin 9\times\sin 11\times\sin 13}$$

$$=\frac{\text{Product of sines of R.H. angles}}{\text{Product of sines of L.H. angles}}.$$

Whence :

sum of log sines of R.H. angles = sum of log sines of L.H. angles.

Interlacing triangles. Unless the angular points are subdivided into left- and right-hand angles, it is impossible to apply the rule of log sines. In the case of interlacing triangles, only the parts of the outer angles are considered, and these may be indicated by the usual convention if the figure is traversed, taking one side after the other in the counterclockwise direction.

$$AB=\frac{BC\,.\sin 4}{\sin 1}$$

$$=\frac{CD\,.\sin 6\,.\sin 4}{\sin 3\,.\sin 1}$$

$$=\frac{DA\,.\sin 8\,.\sin 6\,.\sin 4}{\sin 5\,.\sin 3\,.\sin 1}$$

$$=\frac{AB\,.\sin 2\,.\sin 8\,.\sin 6\,.\sin 4}{\sin 7\,.\sin 5\,.\sin 3\,.\sin 1}.$$

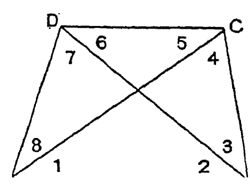


Fig. 168.

Whence : $\sin 1\times\sin 3\times\sin 5\times\sin 7$

$$=\sin 2\times\sin 4\times\sin 6\times\sin 8.$$

Special cases. It follows that a quadrilateral with a single diagonal cannot be checked by the log sine condition. On the other hand, the condition may be introduced in finding angles in precise setting-out work when it is impossible to observe the angle adjacent to one side (the side being impeded) or a diagonal (the diagonal being impeded) (see p. 151).

Notation. (*a*) When the polygons form a series of triangles around a central station, it is advisable to *separate* the outer angles of the figure, so that with the included central angles they define a triangle, the left- and right-hand angles being adjacent to an outer side and best subscribed respectively with *odd* numerals on the left and even on the right. It is also convenient to number the central angles successively first.

(*b*) In the case of quadrilaterals subdivided into triangles without a central station, it is necessary, as stated above, to divide the outer angles into two parts and style these adjacent partial angles *left* and *right*, as, for example, $A_2 + A_1 = A$, the right-hand part taking the odd and the left the even number, in order to be consistent with the notation of (*a*).

In general, the use of numerals alone is more expeditious than subscribing the numbers to letters, as A_1, B_2, C_3, etc.

Limitations. Without some further condition the process is open to an indefinite number of solutions, and the application is either by trial and error or by some conventional artifice leading to direct solution. A fourth condition is imposed by the theory of least squares in that the sum of the squares of the errors shall be a mimimum. Solution on these rational lines is laborious; and in smaller surveys the otherwise conventional treatment may be justified, particularly if the sides do not exceed two miles in length.

(*a*) **Conventional methods.** (1) *Trial and error.* The following considerations may assist in adjustment by this method:

(i) Corrections applied to any triangle hold good for all the figures of which it is a part.

(ii) Apex or central sum open to correction even if it be 360°, as often will be the case, since all uncorrected angles are liable to error.

(iii) Correct small angles in log sines rather than large, since this produces smaller changes in the observed angles, the sines changing more rapidly in small angles.

(iv) Correct large angles in correcting the error in the summation of angles, the differences of the log sines of large angles being small.

(2) *Simplified method of equal shifts.* (i) Apply one-third of each triangle error b_1, b_2, etc., appropriately to each angle, central right and left, so that b_1', b_2', etc., are equal and of opposite sign to the triangle errors.

(ii) If the apex sum is not 360°, distribute the resulting error a' equally among the three angles of each triangle, making the error zero in both the apex and triangle sums, thus giving total correction b_1'', b_2'', etc., with equal right- and left-angle corrections β_1, β_2, etc.

(iii) Assuming that D is the difference in the log sine sums, taken conveniently as D_l, write the expression for the angle shift so that the left-hand angles take the odd numbers in the log sine differences per second δ_1, δ_2; etc. Thus:

$$x = \pm\frac{\beta_1(\delta_1-\delta_2)+\beta_2(\delta_3-\delta_4)+\beta_3(\delta_5-\delta_6)}{\Sigma\delta}+D_l,$$

where D_l may be plus or minus and β_1, β_2, etc., have appropriate signs.

(iv) Subtract the correction x algebraically from β_1, β_2, etc., for the final L.H. angle corrections, and add it accordingly to β_1, β_2, etc., for the final R.H. angle corrections.

Theoretically, this adjustment is no more correct than any one of the final corrections by trial and error.

(*b*) **Adjustment by least squares.** Consider the simplest case; that of a triangle with a central station, as in Fig. 169, where L.H. angles are denoted by odd numbers and R.H. angles by even numbers.

Let e_1+e_2+, etc., be the errors with opposite signs to give corrections and Δ_1, Δ_2, Δ_3, etc., the log sine differences per 1″.

Condition (1) $\quad e_1+e_2+e_3=\pm a.$ ....(1)

Condition (2) $\quad e_1+e_4+e_5=\pm b_1.$ ....(2)

,, ,, $\quad e_2+e_6+e_7=\pm b_2.$ ....(3)

,, ,, $\quad e_3+e_8+e_9=\pm b_3.$ ....(4)

Condition (3) $\quad \Sigma\log\sin R=\Sigma\log\sin L,$

which leads to the equation of log sine differences

$$-\Delta_4e_4+\Delta_5e_5-\Delta_6e_6+\Delta_7e_7-\Delta_8e_8+\Delta_9e_9=D_l. \quad\text{..............(5)}$$

By the theory of least squares, $\Sigma_1^9 e^2$ is to be a minimum. ....(6)

Differentiating the six expressions,

I. $de_1+de_2+de_3=0.$ ....(ζ)

II. $de_1+de_4+de_5=0.$ ....(λ)

III. $de_2+de_7+de_6=0.$ ....(μ)

IV. $de_3+de_8+de_9=0.$ ....(ν)

V. $-\Delta_4de_4+\Delta_5de_5-\Delta_6de_6+\Delta_7de_7-\Delta_8de_8+\Delta_9de_9=0.$ ...(η)

VI. $\Sigma_1^9 e\,.\,de=0.$

Multiplying equations I, II, III, IV, V, by $-\zeta$, $-\lambda$, $-\mu$, $-\nu$, and $-\eta$ respectively and adding to Equation VI:

$$\begin{aligned}&de_1(e_1-\zeta-\lambda)+de_2(e_2-\zeta-\mu)+de_3(e_3-\zeta-\nu)\\&\quad+de_4(e_4-\lambda+\Delta_4\eta)+de_5(e_5-\lambda-\Delta_5\eta)+de_6(e_6-\mu+\Delta_6\eta)\\&\quad+de_7(e_7-\mu-\Delta_7\eta)+de_8(e_8-\nu+\Delta_8\eta)+de_9(e_9-\nu-\Delta_9\eta)=0.\end{aligned}$$

Since each bracketed expression is equal to 0,

$$e_1=\zeta+\lambda;\ e_2=\zeta+\mu;\ e_3=\zeta+\nu;\ e_4=\lambda-\Delta_4\eta;\ e_5=\lambda+\Delta_5\eta;$$
$$e_6=\mu-\Delta_6\eta;\ e_7=\mu+\Delta_7\eta;\ e_8=\nu-\Delta_8\eta;\ e_9=\nu+\Delta_9\eta.$$

Substituting these values of e in the original equations, (1) to (4),

(1′) $3\zeta+\lambda+\mu+\nu=\pm a$; (2′)

(3′) $\zeta+3\mu-(\Delta_6-\Delta_7)\eta=\pm b_2$; (4′)

Now from equations (1′) to (4′) it is possible to eliminate ζ in terms of η and to find λ, μ, and ν in terms of η.

If these values of λ, μ, and ν be substituted in the equations for e_4 to e_9, then the values of e_4 to e_9 may be found in terms of η, giving (say) equations (iv) to (ix).

Then η may be found by substituting these values of e in equation V.

Whence, on substituting for η in equations (iv) to (ix), the values of $e_4, e_5, \dots e_9$ may be found.

Finally, e_1, e_2, and e_3 may be found from equations (2), (3), and (4), and checked by the fact that $e_1=\zeta+\lambda$; $e_2=\zeta+\mu$; and $e_3=\zeta+\nu$.

TRIANGLES WITH CENTRAL STATIONS

The following are the mean observed angles of a triangle *ABC* which was divided into three triangles by lines to the vertex stations from a central station *O*, the angles designated "*R*" and "*L*" being respectively to the *right* or *left* of an observer when traversing the figure and constantly facing the station *O*.

(The numerals in the first column suggest a convenient order for the individual angles when the stations *A*, *B*, *C*, are recorded in the clockwise direction.)

Determine the most probable values of the corrections to the nine angles in accordance with *three* fundamental equations of condition, assuming that the observations are of equal weight.

N.B.—*Full credit will be given only if the solution is based on the Theory of Least Squares.*

Angles		Triangles		
		AOB	*BOC*	*COA*
1 2 3	Central	112° 10′ 45″	119° 49′ 43″	127° 59′ 33″
5 7 9	Left	33° 42′ 44″	32° 03′ 20″	24° 29′ 24″
4 6 8	Right	34° 06′ 33″	28° 07′ 02″	27° 30′ 56″
	Log sin *L*	1·7443101	1·7248828	1·6175606
	,, *R*	1·7487859	1·6732763	1·6646320
Diff. per 1″ $\times 10^{-7}$	,, *L*	31·6	33·6	46·2
	,, *R*	31·1	39·4	40·4

(U.L.)

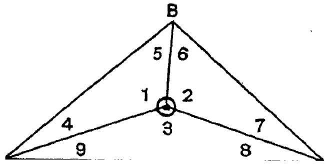


FIG. 109.

Let the corrections applied have the opposite sign to the errors and take the subscripts suggested in the question.

By Condition (i), $e_1 + e_2 + e_3 = -1{\cdot}0''$. ....(1)

By Condition (ii), $e_1 + e_4 + e_5 = -2{\cdot}0''$. ....(2)

$e_2 + e_6 + e_7 = -5{\cdot}0''$. ....(3)

$e_3 + e_8 + e_9 = +7{\cdot}0''$. ....(4)

By Condition (iii), (See (iii), p. 425.)

$$\Sigma \log \sin L = \Sigma \log \sin R\ ;\ D_l - \Sigma e\Delta\,(\log \sin L) + \Sigma e\Delta\,(\log \sin R) = 0.$$

$$-31{\cdot}1e_4 + 31{\cdot}6e_5 - 39{\cdot}4e_6 + 33{\cdot}6e_7 - 40{\cdot}4e_8 + 46{\cdot}2e_9 = +593 = D_l. \quad \text{...(5)}$$

By the theory of least squares, $y = \Sigma_1^9 e^2$ must be a minimum. ......(6)

I. $de_1 + de_2 + de_3 = 0$. ....(ζ)

II. $de_1 + de_4 + de_5 = 0$. ....(λ)

III. $de_2 + de_7 + de_6 = 0$. ....(μ)

IV. $de_3 + de_8 + de_9 = 0$. ....(ν)

V. $-31{\cdot}1de_4 + 31{\cdot}6de_5 - 39{\cdot}4de_6 + 33{\cdot}6de_7 - 40{\cdot}4de_8 + 46{\cdot}2de_9 = 0$. ...($\eta$)

VI. $\Sigma_1^9\, e\,.\,de = 0$.

Adjustment by Least Squares. Multiplying equations I, II, III, IV, and V by ζ, λ, μ, ν, and η respectively, and subtracting from equation VI:

$$\begin{aligned} de_1(e_1 - \zeta - \lambda) &+ de_2(e_2 - \zeta - \mu) + de_3(e_3 - \zeta - \nu) + de_4(e_4 - \lambda + 31{\cdot}1\eta) \\ &+ de_5(e_5 - \lambda - 31{\cdot}6\eta) + de_6(e_6 - \mu + 39{\cdot}4\eta) + de_7(e_7 - \mu - 33{\cdot}6\eta) \\ &+ de_8(e_8 - \nu + 40{\cdot}4\eta) + de_9(e_9 - \nu - 46{\cdot}2\eta). \end{aligned}$$

Since each bracketed expression is equal to 0,

$$e_1 = \zeta + \lambda\ ; \qquad e_2 = \zeta + \mu\ ; \qquad e_3 = \zeta + \nu\ ;$$

$$e_4 = \lambda - 31{\cdot}1\eta\ ; \qquad e_5 = \lambda + 31{\cdot}6\eta\ ; \qquad e_6 = \mu - 39{\cdot}4\eta\ ;$$

$$e_7 = \mu + 33{\cdot}6\eta\ ; \qquad e_8 = \nu - 40{\cdot}4\eta\ ; \qquad e_9 = \nu + 46{\cdot}2\eta.$$

Substituting these values of e in the original equations (1) to (4),

(1′) $3\zeta + \lambda + \mu + \nu = -1$; (2′) $\zeta + 3\lambda + 0{\cdot}5\eta = -2$;

(3′) $\zeta + 3\mu - 5{\cdot}8\eta = -5$; (4′) $\zeta + 3\nu + 5{\cdot}8\eta = +7$.

Adding (2′), (3′), and (4′),

$$3\zeta+3(\lambda+\mu+\nu)+0{\cdot}5\eta=0. \quad \ldots.$$

Subtracting (1′) from equation (5′),

$$2(\lambda+\mu+\nu)+0{\cdot}5\eta=1, \quad \text{or} \quad (\lambda+\mu+\nu)=-0{\cdot}25\eta+0{\cdot}5.$$

Also from (1′), $3\zeta-0{\cdot}25\eta+0{\cdot}5+1=0$; $\zeta=-0{\cdot}5+0{\cdot}08\eta$.

Substituting this last value in (2′), (3′) and (4′),

$$3\lambda+\zeta+0{\cdot}5\eta=-2\ ; \ 3\lambda=0{\cdot}5-0{\cdot}08\eta-0{\cdot}5\eta-2=-1{\cdot}5-0{\cdot}58\eta\ ;$$
$$\lambda=-0{\cdot}5-0{\cdot}19\eta.$$

$$3\mu+\zeta-5{\cdot}8\eta=-5\ ; \ 3\mu=0{\cdot}5-0{\cdot}08\eta+5{\cdot}8\eta-5=-4{\cdot}5+5{\cdot}72\eta\ ;$$
$$\mu=-1{\cdot}5+1{\cdot}91\eta.$$

$$3\nu+\zeta+5{\cdot}8\eta=+7\ ; \ 3\nu=0{\cdot}5-0{\cdot}08\eta+5{\cdot}8\eta+7=7{\cdot}5-5{\cdot}88\eta\ ;$$
$$\nu=2{\cdot}5-1{\cdot}96\eta.$$

Inserting these values of λ, μ, and ν in terms of η in the equations for e_4 to e_9 :

$$\text{(iv)} \quad e_4=\lambda-31{\cdot}5\eta=-0{\cdot}5-0{\cdot}19\eta-31{\cdot}1\eta=-0{\cdot}5-31{\cdot}29\eta,$$
$$\text{(v)} \quad e_5=\lambda+31{\cdot}6\eta=-0{\cdot}5-0{\cdot}19\eta+31{\cdot}6\eta=-0{\cdot}5+31{\cdot}41\eta,$$
$$\text{(vi)} \quad e_6=\mu-39{\cdot}4\eta=-1{\cdot}5+1{\cdot}91\eta-39{\cdot}4\eta=-1{\cdot}5-37{\cdot}49\eta,$$
$$\text{(vii)} \quad e_7=\mu+33{\cdot}6\eta=-1{\cdot}5+1{\cdot}91\eta+33{\cdot}6\eta=-1{\cdot}5+35{\cdot}51\eta,$$
$$\text{(viii)} \quad e_8=\nu-40{\cdot}4\eta=+2{\cdot}5-1{\cdot}96\eta+40{\cdot}4\eta=+2{\cdot}5-42{\cdot}36\eta,$$
$$\text{(ix)} \quad e_9=\nu+46{\cdot}2\eta=+2{\cdot}5-1{\cdot}96\eta+46{\cdot}2\eta=+2{\cdot}5+44{\cdot}24\eta.$$

Substituting these values of e in equation (5) :

$$+31{\cdot}1(0{\cdot}5+31{\cdot}29\eta)-31{\cdot}6(0{\cdot}5-31{\cdot}41\eta)+39{\cdot}4(1{\cdot}5+37{\cdot}49\eta)$$
$$-33{\cdot}6(1{\cdot}5-35{\cdot}51\eta)-40{\cdot}4(2{\cdot}5-42{\cdot}36\eta)+46{\cdot}2(2{\cdot}5+44{\cdot}24\eta)=593$$
$$+23+8361\eta=593\ ; \quad \text{or} \quad \eta=\tfrac{570}{8361}=0{\cdot}0682.$$

Substituting for η in equations (iv) to (ix) :

$$e_4=-0{\cdot}5-31{\cdot}29\times0{\cdot}0682=-0{\cdot}5-2{\cdot}13=-2{\cdot}63''.$$
$$e_5=-0{\cdot}5+31{\cdot}42\times0{\cdot}0682=-0{\cdot}5+2{\cdot}14=+1{\cdot}64''.$$
$$e_6=-1{\cdot}5-37{\cdot}49\times0{\cdot}0682=-1{\cdot}5-2{\cdot}56=-4{\cdot}06''.$$
$$e_7=-1{\cdot}5+35{\cdot}51\times0{\cdot}0682=-1{\cdot}5+2{\cdot}42=+0{\cdot}92''.$$
$$e_8=+2{\cdot}5-42{\cdot}36\times0{\cdot}0682=+2{\cdot}5-2{\cdot}89=-0{\cdot}39''.$$
$$e_9=+2{\cdot}5+44{\cdot}24\times0{\cdot}0682=+2{\cdot}5+3{\cdot}02=+5{\cdot}52''.$$

The values of e_1, e_2, and e_3 may be determined by equations (2), (3), and (4) as follows :

$$e_1 = -2-(e_4+e_5) = -2+0{\cdot}99 = -1{\cdot}01'' ;$$
$$e_2 = -5-(e_6+e_7) = -5+3{\cdot}14 = -1{\cdot}86'' ;$$
$$e_3 = 7-(e_8+e_9) = +7-5{\cdot}13 = +1{\cdot}87''.$$

A check on the evaluation will be found by determining ζ, λ, μ, and ν :

$$\zeta = -0{\cdot}5+0{\cdot}08\times 0{\cdot}0682 = -0{\cdot}5+0{\cdot}01 = -0{\cdot}49.$$
$$\lambda = -0{\cdot}5-0{\cdot}19\times \quad ,, \quad = -0{\cdot}5-0{\cdot}01 = -0{\cdot}51.$$
$$\mu = -1{\cdot}5+1{\cdot}91\times \quad ,, \quad = -1{\cdot}5+0{\cdot}13 = -1{\cdot}37.$$
$$\nu = +2{\cdot}5-1{\cdot}96\times \quad ,, \quad = +2{\cdot}5-0{\cdot}13 = +2{\cdot}37.$$

Whence :
$$e_1 = \zeta+\lambda = -0{\cdot}49-0{\cdot}51 = -1{\cdot}00'' ;$$
$$e_2 = \zeta+\mu = -0{\cdot}49-1{\cdot}37 = -1{\cdot}86'' ;$$
$$e_3 = \zeta+\nu = -0{\cdot}49+2{\cdot}37 = +1{\cdot}88''.$$

Solution by equal shifts. Here the triangle errors are

$$b_1 = -2'' ; \quad b_2 = -5'' ; \quad b_3 = +7''.$$

Applying one-third of each of these errors to the central, left- and right-hand angles :

Centre	Left	Right	
$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$b_1' = -2''$
$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	$b_2' = -5''$
$+\frac{7}{3}$	$+\frac{7}{3}$	$+\frac{7}{3}$	$b_3' = +7''$

Since the apex sum is still 360° 00′ 01″, $-\frac{1}{3}''$ is to be applied to ach central angle and $+\frac{1}{6}''$ to each left- and right-hand angle ; thus :

$-\frac{3}{3}$	$-\frac{3}{6}$	β_1	$-\frac{3}{6}$	$b_1'' = -2''$
$-\frac{6}{3}$	$-\frac{9}{6}$	β_2	$-\frac{9}{6}$	$b_2'' = -5''$
$+\frac{6}{3}$	$+\frac{15}{6}$	β_3	$+\frac{15}{6}$	$b_3'' = +7''$

$$x = \pm\frac{\beta_1(\delta_1-\delta_2)+\beta_2(\delta_3-\delta_4)+\beta_3(\delta_5-\delta_6)+D_l}{\Sigma\delta}$$

$$\pm\frac{\frac{1}{2}(0{\cdot}5)-\frac{3}{2}(-5{\cdot}8)+\frac{5}{2}(5{\cdot}8)+593}{222{\cdot}3} = 2{\cdot}75''.$$

Whence the corrections to L.H. angles : $-3{\cdot}25''$, $-4{\cdot}25''$, $-0{\cdot}25''$.

,, ,, ,, ,, R.H. ,, $+2{\cdot}25''$, $+1{\cdot}25''$, $+5{\cdot}25''$.

QUADRILATERALS WITH INTERLACING DIAGONALS

Example††. The following are the mean observed angles in a quadrilateral $ABCD$, lettered in clockwise order from the south-east station A; numbers, also clockwise from A, being applied to the two parts of the angles at the stations; *even* numbers to *right*-hand and *odd* to *left*-hand partial angles, as designated R and L accordingly in the notes. There is no station at the intersection of the diagonals AC and BD.

Determine the most probable values of the corrections to the *eight* angles by the method of least squares, introducing three *fundamental* equations of condition, and assuming that all the observations are of the same weight.

Angle		BAC / ABD	CBD / ACB	ACD / BDC	BDA / CAD
1, 3, 5, 7	Left	51° 08′ 05″	36° 13′ 20″	61° 45′ 15″	30° 13′ 25″
2, 4, 6, 8	Right	62° 54′ 15″	29° 44′ 25″	52° 17′ 15″	35° 44′ 10″
	log sin L	$\bar{1}$·8913276	$\bar{1}$·7715277	$\bar{1}$·9449390	$\bar{1}$·7018925
	log sin R	$\bar{1}$·9495100	$\bar{1}$·6955422	$\bar{1}$·8982259	$\bar{1}$·7664522
Diff. per 1″	L	16·95	28·73	11·32	36·13
× (10^{-7})	R	10·78	36·85	16·28	29·27

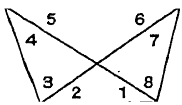


FIG. 170.

Here the corrections with the opposite signs to the errors are:

$$a = -10''; \quad b_1 = +10''; \quad b_2 = -10'',$$

with $\Sigma \log \sin L - \Sigma \log \sin R = 435 \times 10^{-7}$.

The equations of condition are:

(1) $\Sigma\theta = 360°$; (2) $1+2=5+6$;

(3) $3+4=7+8$; (4) $\Sigma \log \sin L = \Sigma \log \sin R$;

(5) Σe^2 a minimum; $e \,.\, de = 0$; where the values of e, appropriately subscribed, are the corrections.

Explicitly, the foregoing equations are:

(1) $e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8 = a = -10''$.

(2) $e_1+e_2 \qquad -e_5-e_6 \qquad = b_1 = +10''$.

(3) $\qquad e_3+e_4 \qquad -e_7-e_8 = b_2 = -10''$.

(4) $e_1\Delta_1 - e_2\Delta_2 + e_3\Delta_3 - e_4\Delta_4 + e_5\Delta_5 - e_6\Delta_6 + e_7\Delta_7 - e_8\Delta_8 = \Sigma e\Delta_L - \Sigma e\Delta_R$
$= 435$,

where Δ_1, Δ_2, etc., are the log sine differences per 1″.

Differentiating, applying the multipliers (indicated in brackets) and adding to $\Sigma_0^8 e\,de$ (p. 425) :

$$\text{I.}\quad de_1+de_2+de_3+de_4+de_5+de_6+de_7+de_8=0 \qquad (-\zeta).$$

$$\text{II.}\quad de_1+de_2-de_5-de_6=0 \qquad (-\lambda).$$

$$\text{III.}\quad de_3+de_4-de_7-de_8=0 \qquad (-\mu).$$

$$\text{IV.}\quad \Delta_1 de_1-\Delta_2 de_2+\Delta_3 de_3-\Delta_4 de_4+\Delta_5 de_5-\Delta_6 de_6+\Delta_7 de_7-\Delta_8 de_8=0 \ (-\eta).$$

Whence : (i) $e_1=\zeta+\lambda+\Delta_1\eta$; (v) $e_5=\zeta-\lambda+\Delta_5\eta$;

(ii) $e_2=\zeta+\lambda-\Delta_2\eta$; (vi) $e_6=\zeta-\lambda-\Delta_6\eta$;

(iii) $e_3=\zeta+\mu+\Delta_3\eta$; (vii) $e_7=\zeta-\mu+\Delta_7\eta$;

(iv) $e_4=\zeta+\mu-\Delta_4\eta$; (viii) $e_8=\zeta-\mu-\Delta_8\eta$.

Substituting these values of e in the original equations, (1) to (4) :

(1′) $8\zeta+\eta(\Delta_1-\Delta_2+\Delta_3-\Delta_4+\Delta_5-\Delta_6+\Delta_7-\Delta_8)=a$,

or $8\zeta-0{\cdot}05\eta=-10''$. $\zeta=-1{\cdot}25+0{\cdot}00625\eta$.

(2′) $4\lambda+\eta(\Delta_1-\Delta_2-\Delta_5+\Delta_6)=b_1$,

or $4\lambda+11{\cdot}13\eta=+10''$. $\lambda=2{\cdot}5-2{\cdot}7825\eta$.

(3′) $4\mu+\eta(\Delta_3-\Delta_4-\Delta_7+\Delta_8)=b_2$,

or $4\mu-14{\cdot}98\eta=-10''$. $\mu=-2{\cdot}5+3{\cdot}7450\eta$.

(4′) $\zeta(\Delta_1-\Delta_2+\Delta_3-\Delta_4+\Delta_5-\Delta_6+\Delta_7-\Delta_8)+\lambda(\Delta_1-\Delta_2-\Delta_5+\Delta_6)$
$+\mu(\Delta_3-\Delta_4-\Delta_7+\Delta_8)+\eta\Sigma(\Delta^2)=\Sigma e\Delta_L-\Sigma e\Delta_R$,

or $0\times\zeta+11\,.\,13\lambda-14\,.\,98\mu+5144\eta=+435$.

Eliminating ζ, λ, and μ by substituting their values in terms of η in (4′) :

$$\eta=+\frac{369{\cdot}72}{5050{\cdot}45}=+0{\cdot}07321.$$

Substituting for η in (1′), (2′), and (3′),

$$\zeta=-1{\cdot}2495\ ;\quad \lambda=+2{\cdot}2963\ ;\quad \mu=-2{\cdot}2249.$$

Inserting these four values in equations (i) to (viii) :

$e_1=+2{\cdot}288''$; $e_3=-1{\cdot}373''$; $e_5=-2{\cdot}717''$; $e_7=+3{\cdot}620''$;

$e_2=+0{\cdot}258''$; $e_4=-6{\cdot}173''$; $e_6=-4{\cdot}737''$; $e_8=-1{\cdot}167''$.

Checks : $\Sigma e=-10{\cdot}001''$;

,, $e_1+e_2-e_5-e_6=+10{\cdot}000''$;

,, $e_3+e_4-e_7-e_8=-10{\cdot}001''$;

,, $\Sigma e\Delta_L-\Sigma e\Delta_R=+435{\cdot}64$.

QUESTION ON ARTICLE 4

1†. Detail the process of adjustment in accordance with the theory of least squares in the case of a triangle subdivided into three triangles by means of a central station, the observations at the four stations being of equal weight. (U.L.)

2. What are the three equations for a perfect polygon which are employed if its angles are to be adjusted simply and quickly?

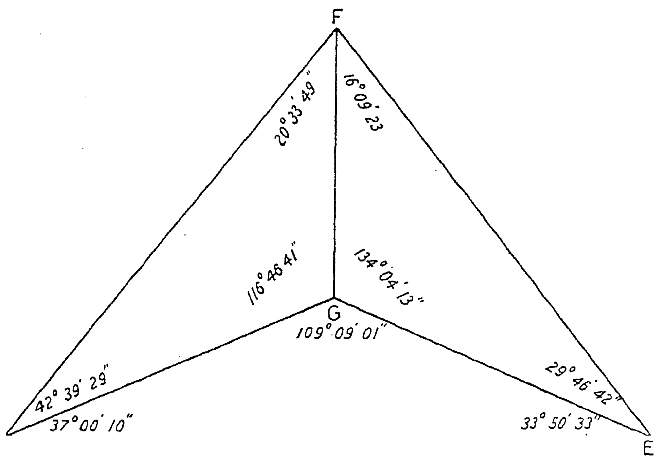
Employ these equations to adjust the angles of the triangle *DEF* with a central point *G* in the accompanying sketch.

Given :

Angle	*L* sine	*L* sine difference for 1″
37° 00′ 10″	9·7794910	28
29° 46′ 42″	9·6960467	37
20° 33′ 49″	9·5456127	56
33° 50′ 33″	9·7457865	31
16° 09′ 23″	9·4444509	73
42° 39′ 29″	9·8309872	23

(U.B.)

[Corrections to above angles in order by equal shifts :
$+6{\cdot}56''$; $-4{\cdot}18''$; $+1{\cdot}57''$; $+2{\cdot}32''$; $-9{\cdot}02''$; $-2{\cdot}67''$; $x=-2{\cdot}12''$; the central angles being
116° 46′ 43·11″; 134° 04′ 08·7″; 109° 09′ 08·11″.]



3. What is meant by the term Condition Equation?

Explain what types of condition equations occur in a free triangulation net, and why these conditions have to be satisfied.

Show that $\log_{10}\sin(\alpha+x'')=\log_{10}\sin\alpha+x''$ diff. for 1″ at $\log_{10}\sin\alpha+\ldots$, and verify by calculation and log tables for $\alpha=27°$ and $x=5''$. Explain how this is utilised in the evaluation of a side equation. (U.C.T.)

ARTICLE 5: ERRORS OF TACHEOMETRY

The aim of the present article is to investigate the magnitudes of the errors likely to occur in tacheometrical observations when the errors arising from imperfect adjustment and manipulation are avoided and the systematic errors arising from defects in the staff, interval, or tangent scale are controlled or eliminated, leaving the accidental errors preponderant.

I. **Subtense system.** Here the errors may originate from two sources:

(*a*) From δs, the error in reading the intercept s on a vertical staff. This appears to consist of a constant term and increases further with the distance, but may in general be expressed by $eD \,.\, \sec^2\alpha$, where α is the vertical angle, and e a fixed value depending upon the telescope, the sharpness of the webs, and the dividing of the staff.

Thus $$\propto s \propto \frac{D}{100\cos^2\alpha} = eD\sec^2\alpha.$$

When the error is expressed as δs in D ft. horizontally, say 0·01 ft. in 400 ft., $\delta D_1/D = \frac{1}{400}$ directly, or $e = \frac{1}{40000}$.

(*b*) From angular error, $\delta\alpha$, which may be expressed in minutes, one minute leading to an error of $\frac{1}{3440}$ radian per foot, or $0{\cdot}000291aD$ ft. with a minutes error in a distance of D ft.

Hence the error in the horizontal component D,

(i) $$\left.\begin{aligned} D &= 100s\cos^2\alpha \\ \frac{\delta D_1}{\delta s} &= 100\cos^2\alpha \end{aligned}\right\} \quad \begin{aligned} \delta D_1 &= 100\cos^2\alpha\,\delta s = 100\cos^2\alpha \,.\, \sec^2\alpha \,.\, . \\ &= 100\, e \,.\, D. \end{aligned}$$

(ii) $$\left.\begin{aligned} D &= 100s\cos^2\alpha \\ \frac{\delta D_2}{\delta\alpha} &= -100s\times 2\sin\alpha\cos\alpha = -2V \end{aligned}\right\} \quad \begin{aligned} D &= -2V\delta\alpha = -2D \,.\, \tan\alpha\,\delta\alpha \\ &= 0{\cdot}000291\, aD \times 2\tan\alpha. \end{aligned}$$

Whence the total error:

$$\varDelta D = \pm\sqrt{(\delta D_1)^2 + (\delta D_2)^2} = \pm\sqrt{(100eD)^2 + (0{\cdot}00029aD\times 2\tan\alpha)^2}. \quad \ldots(1)$$

Also the error in the vertical component V:

(i) $$\left.\begin{aligned} V &= 100s\cos\alpha\sin\alpha \\ \frac{\delta V_1}{\delta s} &= 100\cos\alpha\sin\alpha \end{aligned}\right\} \quad \begin{aligned} \delta V_1 &= 100\sin\alpha\cos\alpha \,.\, eD\sec^2\alpha \\ &= 100eD\tan\alpha. \end{aligned}$$

(ii) $$\left.\begin{aligned} V &= 100s\cos\alpha\sin\alpha \\ \delta V_2 &= 100s\cos 2\alpha \,.\, \delta\alpha \end{aligned}\right\} \quad \text{But } 100s = \frac{2V}{\sin 2\alpha}, \text{ and } V = D\tan\alpha$$

$$\delta V_2 = 2V\cot 2\alpha\,\delta\alpha = 2D\tan\alpha\cot 2\alpha\,\delta\alpha$$

Whence on expressing $\delta\alpha$ radians as a minutes :

$$\delta V_2 = (1 - \tan^2\alpha) \times 0{\cdot}00029aD \; ; \quad \ldots(2)$$

and the total error :

$$= \pm\sqrt{(100eD \, . \tan\alpha)^2 + \{0{\cdot}00029aD\,(1 - \tan^2\alpha)\}^2}. \quad \ldots(3)$$

The second term, as expressed by (2), is a useful criterion as to the accuracy of determining vertical components.

II. **Tangential system.** Here the errors may be said to arise from three sources :

(*a*) Error δD_1, due to δs in observing the distance between the vanes, or the intercept as read on a vertical staff.

(*b*), (*c*) Errors δD_2, δD_3, due to errors $\delta\phi$ and $\delta\theta$ in reading the tangents, or in setting to tangents of ϕ and θ, being the respective cases of the Bell-Elliot and Barcena methods.

Error in horizontal distance, D.

Fundamentally, $$D = \frac{s}{\tan\phi - \tan\theta}. \quad \ldots\ldots(4)$$

$$(a)\;\; \delta D_1 = +\frac{s}{\tan\phi - \tan\theta} = +\frac{D}{s}\,\delta s. \quad \ldots\ldots(5)$$

$$(b)\;\; \delta D_2 = -\frac{s \, . \sec^2\phi \, . \, \delta\phi}{(\tan\phi - \tan\theta)^2} = -\frac{D^2}{s}\sec^2\phi \, . \, \delta\phi. \quad \ldots\ldots(6)$$

$$(c)\;\; \delta D_3 = +\frac{s \, . \sec^2\theta \, . \, \delta\theta}{(\tan\phi - \tan\theta)^2} = +\frac{D^2}{s}\sec^2\theta \, . \, \delta\theta. \quad \ldots\ldots(7)$$

Then if δD_1, δD_2, and δD_3 are the probable (m.s. or average) errors, the total error in D is

$$\Delta D = \pm\frac{D^2}{s}\sqrt{\left(\frac{\delta s}{D}\right)^2 + (\sec^4\phi + \sec^4\theta)\,(\delta\theta)^2}, \quad \ldots\ldots(8)$$

since $\delta\phi = \delta\theta$.

Unless the error in holding the staff is appreciable, $\delta s/D$ is likely to be very small ; then

$$\Delta D = \pm\frac{D}{s}\,\frac{a}{3440}\sqrt{\sec^4\phi + \sec^4\theta}, \quad \ldots\ldots(9)$$

while as a first approximation $\sqrt{2\sec^4\phi}$ may be written for $\sqrt{\sec^4\phi + \sec^4\theta}$.

Error in vertical component, V.

Fundamentally, $$V = \frac{s\tan\theta}{\tan\phi - \tan\theta}. \quad \ldots\ldots(10)$$

$$(a)\ \delta V_1 = +\frac{\tan\theta\,.\,\delta s}{\tan\phi - \tan\theta} = \frac{V}{s}\,\delta s = +\frac{D\tan\theta}{s}\,.\,\delta s. \quad \ldots\ldots\ldots\ldots(11)$$

$$(b)\ \delta V_2 = -\frac{s\,.\,\sec^2\phi\tan\theta}{(\tan\phi - \tan\theta)^2}\cdot\delta\phi = -\frac{D^2}{s}\sec^2\phi\,.\tan\theta\,\delta\phi. \quad \ldots\ldots(12)$$

$$(c)\ \delta V_3 = +\frac{s\sec^2\theta\tan\phi}{(\tan\phi - \tan\theta)^2}\cdot\delta\theta = +\frac{D^2}{s}\sec^2\theta\tan\phi\,\delta\theta. \quad \ldots\ldots(13)$$

Whence the probable (m.s. or average) error is

$$\Delta V = \pm\frac{D}{s}\sqrt{\tan^2\theta(\delta s)^2 + D^2(\sec^4\phi\tan^2\theta + \sec^4\theta\tan^2\phi)(\delta\theta)^2}. \quad \ldots(14)$$

Since, as before, δs is likely to be very small,

$$\Delta V = \pm\frac{D}{s}(\delta\theta)\sqrt{\sec^4\phi\tan^2\theta + \sec^4\theta\tan^2\phi}, \quad \ldots\ldots\ldots\ldots\ldots(15)$$

where $\delta\theta = 0{\cdot}00029$ radians per 1′.

xample††. The following levels were taken with an anallatic tacheometer in which the multiplier was 100.

Station	Height of trunnion	Staff at	Subtense wires			Vertical angle + elevation − depression
			L	M	U	
I	4·40	B.M. 68·4	7·60	4·99	2·38	−6° 30′
		II	6·84	4·64	2·44	+9° 30′
II	4·65	III	8·16	5·85	3·54	+6° 30′
III	4·70	IV	6·42	4·82	3·22	−8° 24′
IV	4·52	V	8·08	5·64	3·20	−9° 30′
V	4·44	B.M. 107·3	6·86	4·96	3·06	−2° 20′

Calculate the elevations of Stations I, II, III, IV, and V and find the error of closure on the B.M. 107·3. Also determine the probable errors in each of these elevations, assuming that an error of 20″ may occur in any vertical angle, and that the error of reading the intercepts is 1 in 333, varying directly with the sight length. (U.L.)

Reduced elevations: 127·66, 199·05, 249·78, 203·42, 122·85, with closing error −0·40 ft.

$$D = 100s\cos^2\alpha.$$

If error of $\dfrac{s}{333}$ occurs in intercept s, then from this source,

$$dD_1 = \pm D/333 \quad \text{and} \quad dV_1 = \pm 30D\tan\alpha\times 10^{-4}.$$

$V = 100s\sin\alpha\cos\alpha$; $dV_2 = \pm 100s(\cos^2\alpha - \sin^2\alpha)\,d\alpha$, with $d\alpha$ in radians,

$= 100s\cos^2\alpha(1 - \tan^2\alpha)\,d\alpha\times 0{\cdot}485\times 10^{-5}$, with $d\alpha$ in seconds.

When $\alpha = 20''$, $dV_2 = \pm D \times 10^{-4} \sqrt{0{\cdot}94(1 - \tan^2\alpha)^2}$,

where D is the horizontal distance.

Whence the probable error from both sources :

$$\times 10^{-4} \sqrt{900 \tan^2\alpha + 0{\cdot}94(1 - \tan^2\alpha)^2}.$$

Then if ΔV_1, ΔV_2, ΔV_3, etc., be the total probable errors in each elevation, $\Delta V_1 = dV_1$; $\Delta V_2 = \pm\sqrt{(dV_1)^2 + (dV_2)^2}$, etc., following the square roots of the sums of the squares of successive errors, as tabulated hereafter :

$$dV = \pm 0{\cdot}183 ; \quad \pm 0{\cdot}218 ; \quad \pm 0{\cdot}162 ; \quad \pm 0{\cdot}142 ; \quad \pm 0{\cdot}242 ; \quad \pm 0{\cdot}059.$$

$$\Delta V = \sqrt{\Sigma (dV)^2} = \pm 0{\cdot}183 ; \pm 0{\cdot}285 ; \pm 0{\cdot}328 ; \pm 0{\cdot}357 ; \pm 0{\cdot}431 ; \pm 0{\cdot}436.$$

Example†. Barcena's method was attempted with a theodolite provided with tangent divisions of 0·01 on the vertical circle, the error of setting to one of these divisions being $\frac{1}{2}'$ for both of the involved vertical angles.

Determine the error in the horizontal distance when an observed staff intercept of 4·22 ft. followed from vertical angles with tangent values of 0·1100 and 0·1000 respectively.

$$D = \pm \frac{(422)^2}{4{\cdot}22} \sqrt{(1 + \tan^2\phi)^2 + (1 + \tan^2\theta)^2} \,.\, \tfrac{1}{2}(0{\cdot}00029) \text{ ft.}$$

$$= \pm 42200 \sqrt{(1{\cdot}0121)^2 + (1{\cdot}0100)^2} \,.\, \tfrac{1}{2}(0{\cdot}00029) = 8{\cdot}75 \text{ ft.}$$

Example†. Derive an equation for the determination of horizontal distances in the tangential system of tacheometry in which the tangents of vertical angles are observed for a constant intercept on a vertical staff. Also derive an expression for the probable error in horizontal distances when error is assumed to arise only from the observation of vertical angles, the error in the intercept being negligible.

Assuming that the probable error in observing the tangents of vertical angles is 10 seconds, state the probable error in a horizontal distance of 800 ft. for a staff intercept of 10 ft. and a maximum vertical angle of 15°. (U.L.)

Since $\delta\phi = \delta\theta$, the error $\Delta D = \pm \dfrac{D^2}{s} \sqrt{(\sec^4\phi + \sec^4\theta)(\delta\theta)^2}$.

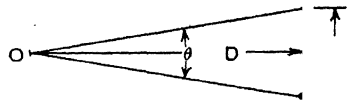
Now $$\tan\phi - \tan\theta = 10/800,$$

and $$\tan\theta = \tan 15° - 0{\cdot}0125 = 0{\cdot}2554492 ; \quad \theta = 14° \; 19{\cdot}78'$$

$$\frac{(800)^2}{10} \sqrt{} \qquad 13477(0{\cdot}0000485)$$

$$= 0{\cdot}568 \text{ ft. or 1 in 1410.}$$

Example†. A subtense method of tacheometry consists in observing n times the angle subtended by a horizontal base s ft. in length.



171.

Derive an expression for the probable error ΔD in determining horizontal distances D when the combined error in sighting both ends of the base is $d\theta$.

Calculate accordingly the error in a horizontal distance of 600 ft. when the angle subtended is measured five times and the combined error in sighting the targets of a 3-ft. base is 10″. (U.L.)

$$D = \tfrac{1}{2}s \,.\, \cot \tfrac{1}{2}\theta = \frac{\tfrac{1}{2}s}{\tan \tfrac{1}{2}\theta};$$

since θ is small, $$D = \frac{\tfrac{1}{2}s}{\tfrac{1}{2}\theta \tan 1''} = \frac{s}{\theta \tan 1''}, \text{ with } \theta \text{ in sec. } \qquad \text{.........(1)}$$

Calculation may be expedited by writing (1) as follows:

$$\log D = \log s - \log \theta'' - \bar{6}{\cdot}6855749.$$

Since the error $d\theta$ incorporates (*a*) bisecting and sighting, and (*b*) vernier reading, the problem is simplified, and

$$\frac{dD}{d\theta} = -\frac{s}{\theta^2 \tan 1''}, \text{ and the probable error } dD = \pm \frac{s}{\theta^2 \tan 1''}\, d\theta.$$

But $s = D \,.\, \theta \tan 1''$ $$\therefore\ dD = \pm \frac{D}{\theta} d\theta. \qquad \text{............(2)}$$

Here $\tan \theta = \dfrac{3}{600}$; $d\theta = \dfrac{10''}{\sqrt{5}}$; and, by logarithmic calculation in (2), $D = 2{\cdot}602$ ft., or 1 in 230.

QUESTIONS ON ARTICLE 5

1†. Derive expressions for the probable errors of determining horizontal distances D and vertical components V with an anallatic subtense tacheometer.

Account only need be taken of the errors of reading the staff intercepts s and the vertical angles α, the former presumably following a linear law. (U.L.)

2†. Derive a working rule for the probable errors in horizontal distances and vertical components observed with an anallatic tacheometer which shows a probable error of $\tfrac{1}{2}'$ in observing vertical angles and a probable error of 0·01 ft. in the intercepts observed at a distance of 250 ft. on a vertical staff, this error varying with the distance. (U.L.)

[$D/1000\sqrt{16 + 0{\cdot}09 \tan^2\alpha}$; $D/1000\sqrt{16 \tan^2\alpha + 0{\cdot}0225(1 - \tan^2\alpha)^2}$]

3†. Derive an equation for the determination of horizontal distances in Barcena's tangential system of tacheometry, the staff intercepts corresponding to tangent differences of 0·01 being observed on a vertical staff.

Determine a working approximation to the probable error D in horizontal distances D, assuming that it is possible to set the verniers of the vertical circle for even tangent differences of 0·01 with a maximum error of half a minute, and that the greatest vertical angles ϕ occurrent will be such that $\tan\phi = 100/D$.

N.B.—The probable error of reading the staff intercepts may be regarded as being negligible. (U.L.)

[$\Delta D = 0{\cdot}0206D(1 + 10^4/D^2)$]

4†. Derive expressions for the probable errors of determining horizontal distances D and vertical components V in the tangential system of tacheometry when account only is taken of the errors of reading the tangents of vertical angles θ and ϕ in sighting a fixed 10-ft. base on a vertical staff. (U.L.)

5†. After running an unsatisfactory line of levels with an anallatic tacheometer the following defects were noted, all other adjustments, etc., being in order, including 100·00 for the multiplier :

(*a*) Eccentricity of vertical circle, $2'$ in range involved.

(*b*) Staff leaning towards instrument, $3°$ out of vertical, on account of defective hinge on staff bubble.

Determine the total error in elevation arising from these two sources in the maximum horizontal distance and vertical angle of 500 ft. and $8° 30'$ respectively. (U.L.)

[(*a*) Accidental error $\pm 0{\cdot}003aD(1 - \tan^2\alpha) = \pm 0{\cdot}293$ ft.
(*b*) Systematic error $-0{\cdot}003aD\tan^2\alpha = -0{\cdot}605$ ft.,
a being the error in minutes in each case.]

6. Compare the accuracies of distance measurement attainable in fixed hair and tangential tacheometry.

In a fixed hair instrument the telescope gives accuracy of staff reading corresponding to $\pm 3''$ of angular error and the multiplying constant is 100. State the corresponding errors in the observed length L of a line.

(*a*) When the instrument is set up over one end and the staff read on the other.

(*b*) When the instrument is set up at the midpoint and the staff read on each end. Assume that the line of sight is level throughout and that there is no additive constant. (U.G.)

1 radian = $206265''$.

[(*a*) $\delta L = \pm kL$; (*b*) $\delta L = \pm \frac{k}{\sqrt{2}}L$, where $k = \frac{1}{688}$.]

ARTICLE 6 : ADJUSTMENT OF TRAVERSE SURVEYS

It seldom happens that the algebraical sums of the latitudes and departures of closed traverses are zero. As a rule they exhibit errors, the limits of which are fairly well established. If, however, these errors exceed the permissible limits for calculation or plotting, the traverse requires to be adjusted. Normally, errors appreciable in plotting indicate work containing errors exceeding the permissible errors of the field.

Possibly no surveying problem introduces such diversity of opinion or uncertainty of thought as the commonplace operation of adjusting traverse surveys, the treatment frequently being regarded as an indefinite means to a definite end.

Primarily the methods may be styled *rational* or *conventional* according as the methods are based upon the theory of least squares, or are merely artifices inducing closure, either as a polygon, or a series of courses between triangulation stations.

Best known of all the rational methods is that due to Dr. Nathaniel Bowditch (1807), styled the **compass rule** in American text-books, either on account of its association with the then prime surveying instrument, or to distinguish it from the conventional **transit rule**, or method of proportional co-ordinates. There appears at present to be a tendency to advance what may be styled **orthomorphic methods** on the grounds that the linear errors are normally inordinately higher than those arising from angular measurement. Accordingly it is assumed that *no* error exists in the individual angles, the adjustment being made solely in the sides, sometimes after the total error has been distributed among the angles and bearings. Among these methods may be cited the rational rules of Crandall * and Rappleye †, and the conventional method due to Prof. Ormsby ‡, the graphical axis method being a geometrical artifice. Incidentally, the best known graphical method is rational in that it is based upon Bowditch's method, questionable as the fundamental assumption may be.

The hypothesis that there are no individual angle errors when the condition $(2N-4)\ 90°$ is satisfied is groundless, small as the errors may be in comparison with those arising from linear sources, and while it is possible that the individual accidental errors could have a zero sum, it is possible that the individual values would not be repeated in

* *Geodesy & Least Squares* (Wiley & Sons).

† *Proc. Amer. Soc. Civ. Eng.*, Vol. 55 (1929).

‡ *Surveying*, by Middleton & Chadwick (Spon).

subsequent measurement; a fact that would be evinced if the angles of a polygon were measured (say) five times with theodolites reading to 1′, 20″, and 10″. Further, the fact that the sum of the angle errors, $\Sigma \pm e$, is zero is no criterion that the relevant displacements $\pm \tan 1' \ \Sigma e \, . \, s$ will be zero, for in the hypothetical absence of error in the lengths of the sides, a polygon could assume a number of configurations and still fulfil the $(2N-4)\,90°$ condition.

General theorem of traverse adjustment.* For many years the author has suggested the following method, the notation being generally :

$s_1, s_2, \ldots$, the lengths of successive sides (AB, BC, ...), or s generally, with $\Sigma s = P$, the perimeter.

$\beta_1, \beta_2, \ldots$, etc., the reduced bearings of these sides.

$l_1, l_2, \ldots$ (or $l_{AB}, l_{BC}, \ldots$), and $d_1, d_2, \ldots$ (or d_{AB}, d_{BC}), the relevant corrections in latitude and departure.

E_l and E_d, the total errors in latitude and departure respectively, the actual linear error of closure being $E = \sqrt{E_l^2 + E_d^2}$.

E_c and E_a, the probable errors of linear and angular measurement, as deduced from actual errors or assumed from independent observations, establishing the relation $\pm\sqrt{(E_c^2 + E_a^2)} = \sqrt{(E_l^2 + E_d^2)}$, which is the basis of the method.

e, the error per angle in minutes with the corresponding unit displacement $a = e \, . \tan 1' = 0{\cdot}3 \times 10^{-3}$ conveniently.

In this co-ordination, some assumption must be made for angular errors, also for the extent to which the square root law is justified in linear measurement. On the assumption of $\pm c\sqrt{s}$ for chaining error and $\pm s(\delta\beta)$ for angular displacement, the probable displacement of the extremity B of a line AB will be $\pm\sqrt{(c^2 s + s^2(\delta\beta)^2)}$; and with the theodolite it may be assumed that $\delta\beta$ will be constant over a very considerable range of distance, the reciprocal errors of distant sights and sight bisection balancing out with a reasonable margin for station eccentricity, and thus leaving the other accidental errors to produce a displacement proportional to the distance. Thus, if $\pm e$ is the error of observing a single angle, the linear displacement will be $e \tan 1'$ per unit distance, leading to a total displacement of $\pm\sqrt{(c^2 s + a^2 s^2)}$ compared with Bowditch's $\pm\sqrt{2} \, . \, c\sqrt{s}$.

Now when linear measurements are made with appropriate methods and precautions, the systematic (or cumulative) errors will be largely eliminated, and if sufficient lengths are laid down so that the compensating uncertainty is effective, the accidental errors will control, justifying the use of the square root law $c\sqrt{s}$; but, on the other hand, the conditions of rough and even fair chaining are normally such that the systematic errors are far more influential, and, in general, the linear law appears to obtain as cs, or s/d if chaining ratios are employed. Throughout

* *Engineering*, Aug. 29th, 1939.

this article the terms "coefficient" and "ratio" are used to denote the respective assumptions of the $\sqrt{s}$ and s laws.

Modes of angular measurement. Since the symbol m will occur in the following treatment, it is desirable to differentiate between the two modes of angular measurement.

"*Free needle*" *observations* may be defined as magnetic bearings taken from the needle at all stations with the compass or mining dial, the bearing of each line being independent of the bearings of all other lines, with the result that magnetic interference distorts the configuration of the traverse. *Here m will be unity.*

"*Fixed needle*" *observations* may be defined as those involved in theodolite traverses, whether the bearings are reduced from back angles or deflection angles, or are taken directly as whole circle bearings or reduced bearings from a meridian, true, magnetic, or assumed.

If the magnetic bearings are observed in this manner, the configuration of a traverse will be unaffected by magnetic interference, though the whole will be displaced bodily with respect to the magnetic north. Also when bearings are observed directly, the total error in the polygon will appear as a final reading on the horizontal circle.

In fixed needle work generally, the error of the preceding angle or bearing will be carried forward to the succeeding angles or bearings, following theoretically the square root law. Thus, if $\pm e$ is the error in a single angle or bearing, the error in the first line will be $\pm e$; in the second, $\pm e\sqrt{2}$; in the third, $\pm e\sqrt{3}$ ..., until the total probable error $\pm e\sqrt{N}$ is attained in the closing line. Hence m must be written progressively 1, 2, 3, ... to N, the values being squared in the formulae. Thus the Bowditch rule is theoretically inadmissible to theodolite practice, since the theory does not embody the summing of angular errors peculiar to fixed needle observations.

(*a*) **Theorem with chaining coefficients.** Consider Fig. 172, where an error $c\sqrt{s}$ in chaining displaces the point B to B', while an error $e\sqrt{m}$ in the bearing displaces B ultimately to C, giving a total error BC.

The displacement $B'C$ is proportional to s, and if $e = 1'$,

$$B'C = 0{\cdot}3s\,\sqrt{m} \times 10^{-3}\text{ ft.} = \sqrt{m}\,as.$$

Hence the probable error in the position of B is $\pm\sqrt{(c^2 s + a^2 m s^2)}$.

Let l and d be the rectangular projections of the error BC on the north and east axes, being the respective errors of latitude and departure in the line AB, so that $BC = \sqrt{(l^2 + d^2)}$.

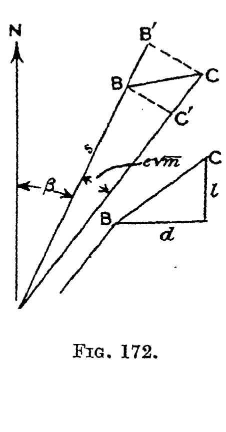


FIG. 172.

Then by the theory of least squares, weighting appropriately by the reciprocal of the combined probable error

$$\Sigma\frac{(l^2+d^2)}{(c^2s+a^2ms^2)} \text{ shall be a minimum, } \quad\text{(i)}$$

and $$\Sigma l = E_l; \quad \text{and} \quad \Sigma d = E_d. \quad\text{(ii)}$$

Differentiating, $$\Sigma\left(\frac{l(\delta l)+d(\delta d)}{c^2s+a^2ms^2}\right)=0. \quad\text{(iv)}$$

$$.\text{(v)} \qquad (\delta d)=0. \quad\text{(vi)}$$

Multiplying (v) and (vi) by $-\eta$ and $-\zeta$ respectively, adding the last three equations, and then equating the coefficients of each δl and δd to zero :

$$\eta=\frac{l_1}{(c^2s_1+a^2ms_1{}^2)}=\frac{l_2}{(c^2s_2+a^2ms_2{}^2)}=\ldots; \quad\text{(vii)}$$

$$\zeta=\frac{d_1}{(c^2s_1+a^2ms_1{}^2)}=\frac{d_2}{(c^2s_2+a^2ms_2{}^2)}=\ldots.$$

Substituting these values in the original equations :

$$\eta=\frac{E_l}{\Sigma(c^2s+a^2ms^2)}; \quad \zeta=\frac{E_d}{\Sigma(c^2s+a^2ms^2)}. \quad\text{(ix), (x)}$$

Whence the corrections in latitude and departure respectively :

$$l_1=\pm\frac{(c^2s_1+a^2ms_1{}^2)}{\Sigma(c^2s+a^2ms^2)}E_l; \quad l_2=\pm\frac{(c^2s_2+a^2ms_2{}^2)}{\Sigma(c^2s+a^2ms^2)}E_l; \quad\text{(1)}$$

$$d_1=\pm\frac{(c^2s_1+a^2ms_1{}^2)}{\Sigma(c^2s+a^2ms^2)}E_d; \quad d_2=\pm\frac{(c^2s_2+a^2ms_2{}^2)}{\Sigma(c^2s+a^2ms^2)}E_d. \quad\text{(2)}$$

(*b*) **Theorem with chaining ratios.** When, as in farm and estate surveying, the errors of chaining are justifiably assumed to follow the linear law, equations (1) and (2) reduce respectively to :

$$l_1=\pm\frac{(c^2s_1{}^2+a^2ms_1{}^2)}{\Sigma(c^2s^2+a^2ms^2)}E_l, \text{ etc.}; \quad d_1=\pm\frac{(c_1{}^2s^2+a^2ms_1{}^2)}{\Sigma(c^2s^2+a^2ms^2)}E_d, \text{ etc.} \quad\text{(3), (4)}$$

Equations (1) to (4) are the fundamental equations of the correlated method of adjustment, and these may be modified in various ways to suit the various modes of angular measurement. A particular advantage of the foregoing rules is that a side or angle, or both, may be given appropriate weight w by merely varying the value of c and (or) a, thus avoiding the dual weighting associated with the Bowditch method.

Methods. The methods most commonly used will now be considered with reference to a common traverse in order to reduce the amount of tabular work. Also the example is taken from the lower class in order to emphasise the processes of adjustment generally, though, on the other hand, fairer comparison would have resulted from higher grade work,

particularly in regard to the correlated method. To a large extent, the grade appears to determine the more amenable method.

Many surveyors make a pre-correctional distribution of angular error, and, doubtless this is warranted in those methods which presuppose no angular error ; but in the correlated method this is not justified theoretically when *actual* angular errors are introduced, though it may be when the angular errors are assumed, as in the simplified applications of this method, apart from the fact that the angle condition $(2N-4)\ 90°$ is always satisfied in the final adjustment.

Example.

Line	Bearing (mag.)	Length (ft.)	Observed latitude	Observed departure
AB	S. 66° 09′ E.	3024·8	−1223·0	+2766·5
BC	N. 38° 20′ E.	1924·6	+1509·7	+1193·7
CD	N. 32° 14′ W.	583·3	+ 493·4	− 311·1
DE	N. 11° 05′ W.	1004·0	+ 985·3	− 193·0
EF	N. 88° 13′ W.	1381·6	+ 43·0	−1380·9
FG	S. 14° 37′ W.	786·7	− 761·2	− 198·5
GA	S. 60° 31′ W.	2144·0	−1055·2	−1866·4
		10849·0	− 8·0	+ 10·3

I. **Bowditch method.**

(*a*) **Consistent errors.** This, the most popular method, is based upon the hypothesis that the linear errors vary as $\sqrt{s}$ and the compass errors e conveniently as $1/\sqrt{s}$. The effect of centring is thus unduly increased in the angular term, which becomes $a\sqrt{s}$, since $m=1$ for the compass in the correlated expressions. Further, the displacements from the two sources are presumably equal with $a=c$. Hence, if $a=c$ and $m=1/s$ are written in the fundamental equations, (1), (2) :

$$l_1=\frac{s_1}{\Sigma s}E_l\,;\quad d_1=\frac{s_1}{\Sigma s}E_d.\ \ldots\ldots\ldots\ldots\ldots\ldots(5),\ (6)$$

Thus the corrections in latitude and departure are in the ratio of the length of the relevant side to the perimeter of the traverse, rendering the method directly amenable to graphical treatment.

$$\text{Or}\quad \frac{\text{Total error in latitude}}{\text{Total length of perimeter}}=\frac{\text{error in latitude of one side}}{\text{length of that side}},$$

$$\frac{\text{Total error in departure}}{\text{Total length of perimeter}}=\frac{\text{error in departure of one side}}{\text{length of that side}}.$$

Also, contrary to a wide-spread belief, (1) the method generally, if anomalously, expresses an ideal unattainable in theodolite traverses, namely, perfect co-ordination of linear and angular errors; and (2) the result would follow from a more general equation in least squares when the angular errors are assumed to be zero. In fact, the rule, under-estimating angular errors, demands linear precision of 1 in $1{\cdot}2 \times 10^6$ with an individual angular error of 10″ in a regular polygon of nine sides, each of 900 ft.

Since the fractions $E_l/\Sigma s$ and $E_d/\Sigma s$ are together common to all the calculations, they may be reduced once and for all as *correction factors.* It then remains to multiply these by the length of the line considered in order to obtain the corrections for the latitude and departure of that line. These corrections carry the signs of the total errors in latitude and departure, and are to be *subtracted algebraically* from the corresponding values in the notes. Otherwise, the corrections may be written with the opposite signs to the errors and *added algebraically.*

Applied to the given example, the correction factors for latitude and departure are $\frac{-8{\cdot}0}{10850}$ and $\frac{+10{\cdot}3}{10850}$ or $-0{\cdot}00074$ and $+0{\cdot}00095$ respectively:

$$\begin{array}{lll}
l_1 = -0{\cdot}00074 \times 3024{\cdot}8 = -2{\cdot}24\,; & \text{also} & d_1 = 0{\cdot}00095 \times 3024{\cdot}8 = 2{\cdot}87 \\
l_2 = -0{\cdot}00074 \times 1924{\cdot}6 = -1{\cdot}42\,; & ,, & d_2 = 0{\cdot}00095 \times 1924{\cdot}6 = 1{\cdot}83 \\
l_3 = -0{\cdot}00074 \times 583{\cdot}3 = -0{\cdot}43\,; & ,, & d_3 = 0{\cdot}00095 \times 583{\cdot}3 = 0{\cdot}55 \\
l_4 = -0{\cdot}00074 \times 1004{\cdot}0 = -0{\cdot}74\,; & ,, & d_4 = 0{\cdot}00095 \times 1004{\cdot}0 = 0{\cdot}95 \\
l_5 = -0{\cdot}00074 \times 1381{\cdot}6 = -1{\cdot}02\,; & ,, & d_5 = 0{\cdot}00095 \times 1381{\cdot}6 = 1{\cdot}31 \\
l_6 = -0{\cdot}00074 \times 786{\cdot}7 = -0{\cdot}58\,; & ,, & d_6 = 0{\cdot}00095 \times 786{\cdot}7 = 0{\cdot}75 \\
l_7 = -0{\cdot}00074 \times 2144{\cdot}0 = -1{\cdot}59\,; & ,, & d_7 = 0{\cdot}00095 \times 2144{\cdot}0 = 2{\cdot}04 \\
\qquad\qquad\qquad\qquad\qquad -8{\cdot}02 & & \qquad\qquad\qquad\qquad +10{\cdot}30
\end{array}$$

Appended are the corrected latitudes and departures, the values, like the original ones, being carried only to one decimal place.

Line	*AB*	*BC*	*CD*	*DE*
Length (ft.):	3024·8	1924·6	583·3	1004·0
Latitude:	−1220·8	+1511·1	+493·8	+ 986·0
Departure:	+2763·6	+1191·9	−311·7	− 194·0

Line	*EF*	*FG*	*GA*	Sums
Length (ft.):	1381·6	786·7	2144·0	10849·0
Latitude:	+ 44·0	−760·6	−1053·6	− 0·1
Departure:	−1382·2	−199·2	−1868·4	0·0

Corrected lengths and bearings. Except in the orthomorphic methods, the effect of correction is tantamount to changing two sides of a right-

angled triangle ; and, in order to complete the arithmetical operation, the line itself, which is the hypotenuse, should be changed accordingly, its new length and bearing being determined. In general, this is a needless refinement, which is unnecessary in plotting by latitudes and departures and, though serving in comparing adjustments by various methods, it applies only to the survey in question, and gives no tangible criterion as to the effects in other surveys. If, however, occasion demands that the new lengths and bearings of lines must be found, it is then merely necessary to reverse the ordinary calculations for latitudes and departures. Thus, the new length of any corrected line is

$$\sqrt{(\text{corrected latitude})^2 + (\text{corrected departure})^2},$$

while the tangent of the new bearing is $\dfrac{\text{corrected departure}}{\text{corrected latitude}}$.

The method is sometimes applied when certain sides are assumed to be correct, the errors and consequent corrections being applied only to the sides presumably affected.

(*b*) **Weighted errors.** The assumption on which the foregoing method is based is that all the measurements are proportionally equal in error, and that the relative accuracy of the bearings and lengths of all the lines is the same. If the field work has been consistent throughout, such procedure is doubtless justified ; but frequently circumstances are such that uniformity is impossible, and it is possible to assert that certain measurements are more accurate than others. In the latter case the usual procedure is as follows :

(1) Adopt one of the traverse lines as a *standard*, and assume that the error e_1 in this line is *unity*.

(2) On this basis estimate the probable error in each of the lines for a distance equal to the standard length, taking into account the circumstances peculiar to each line, both with regard to observing bearings and measuring distances.

(3) Weight each of the lines by multiplying its length by the square of its probable error (or (m.s.e.)^2) for the standard length, $(0{\cdot}5)^2$, $(1{\cdot}0)^2$, $(1{\cdot}5)^2$, $(2{\cdot}0)^2$, ... , as the case may be, the standard error being unity.

(4) Let E_l and E_d be respectively the total errors in latitude and departure in a traverse $ABCD$, etc., l_{AB} and d_{AB} the required corrections for the latitude and departure of the line AB, and e_2, e_3, ... , the probable errors per standard length in the lines in order : then

$$l_{AB} = \frac{(e_1^2 AB)\,E_l}{e_1^2 AB + e_2^2 BC + e_3^2 CD + \dots},$$

and

$$d_{AB}\ \frac{(e_1^2 AB)\,E_d}{e_1^2 AB + e_2^2 BC + e_3^2 CD + \dots}.$$

Sometimes this rule is given with the weight varying merely as the reciprocals of the errors.

(*c*) **Modified compass rule.** If chaining ratios are assumed with $m=1$ and $a=c$, equations (3) and (4) reduce to what is styled the "modified compass rule" in America :

$$l_1=\frac{s_1^2}{\Sigma s^2}E_l; \quad d_1=\frac{s_1^2}{\Sigma s^2}E_d. \quad \text{.................(9), (10)}$$

(*d*) **Correlated compass rule.** The correct rules for compass and mine surveys can be derived by writing $m=1$ in the fundamental equations (1) and (2) or (3) and (4) ; thus :

$$l_1 \text{ (or } d_1)=\frac{(c^2s_1+a^2s_1^2)}{\Sigma(c^2s+a^2s^2)}E_l \text{ (or } E_d); \quad \text{.................(11)}$$

$$l_1 \text{ (or } d_1)=\frac{(c^2s_1^2+a^2s_1^2)}{\Sigma(c^2s^2+a^2s^2)}E_l \text{ (or } E_d). \quad \text{.................(12)}$$

Derived rules. The following rules are based upon the ideal assumption that the probable total displacements due to linear and angular errors should be equal, the assumption of $a=c$ being admissible only to the case of free needle observations.

(*e*) **Bowditch theodolite rule.** Had Bowditch based his rule upon fixed needle observations, this would have been equivalent to writing the theodolite error as $\sqrt{m/s}$, and if m/s is written uniformly for m in the fundamental equation, this will become

$$l_1 \text{ (or } d_1)=\frac{c^2s_1+a^2ms_1}{c^2\Sigma s+a^2\Sigma ms}E_l \text{ (or } E_d).$$

On the further assumption that $c^2=\frac{a^2\Sigma ms}{\Sigma s}$, this equation reduces to the following simple form, which may be styled the "Bowditch Theodolite Rule".

$$l_1\text{(or } d_1)=\tfrac{1}{2}\left\{\frac{s_1}{\Sigma s}+\frac{ms_1}{\Sigma ms}\right\}E_l \text{ (or } E_d). \quad \text{...........(13), (14)}$$

(*f*) **Simplified s and s rules.** In reducing the fundamental equations to the preceding simplified form, a factor k could be applied to the angular equivalent of c, so that

$$c^2=\frac{k^2a^2\Sigma ms^2}{\Sigma s} \quad \text{or} \quad \frac{k^2a^2\Sigma ms^2}{\Sigma s^2},$$

leading respectively to the following rules :

$$l_1 \text{ (or } d_1)=\frac{1}{(k^2+1)}\left\{\frac{k^2s_1}{\Sigma s}+\frac{ms_1^2}{\Sigma ms^2}\right\}E_l \text{ (or } E_d). \quad \text{...........(15)}$$

$$l_1 \text{ (or } d_1)=\frac{1}{(k^2+1)}\left\{\frac{k^2s_1^2}{\Sigma s^2}+\frac{ms_1^2}{\Sigma ms^2}\right\}E_l \text{ (or } E_d). \quad \text{...........(16)}$$

For equal displacements ($k=1$) in a regular polygon of nine sides, each of 900 ft., the ratio of precision $c\sqrt{s}/s$ (or c) would be $\frac{1}{4460}$ to correspond with an individual angular error of 20″; and this fact suggests that there is a wide range of surveys in which the angular term is appreciable and should therefore be appropriately embodied in the correction formulae.

II. Correlated methods.*

Apart from the last fact, the author's correlated rules may commend themselves in many cases by their close adherence to theory and practice and the more intimate knowledge of the fundamentals involved. On the other hand, considerations arise as to the undesirability of superseding the easily applied Bowditch rule or the use of proportional co-ordinates, favoured by many authorities. Besides, the adjustment is comparatively laborious, though the calculations may be facilitated by the use of tables of squares, and if the constants c and a are expressed in terms of 10^{-3} (or 10^{-4}), they will systematically cancel out with $s \times 10^3$, $(ms \times 10^6)^{\frac{1}{2}}$, etc.

The method may be used with (i) actual or (ii) assumed values.

(i) **Actual values.** The actual linear closing error E is known, as also is the observed angular closing error α, which for equal weights is theoretically $e\sqrt{N}$, and may be zero. In practice, however, α is seldom zero with the more precise theodolites, but is occasionally so with instruments reading to 1′ or $\frac{1}{2}$′. In the latter eventuality the problem becomes one of questionable compromise. Thus, if in a total of N angles the error is $\pm e$ per angle and the total error is zero, it might be assumed that n of these angles carry the plus sign and $N-n$ the minus sign, so that

$$e\sqrt{n} - e\sqrt{(N-n)} = 0, \quad \text{or} \quad n = \tfrac{1}{2}N,$$

the error per angle being $\dfrac{1}{\sqrt{2}}$ of its necessarily assumed value e. Incidentally, this forms a criterion for the treatment of traverses by the correlated method, and actual values may be used when α is not less than $e\sqrt{\frac{1}{2}N}$ or is not appreciably in excess of $e\sqrt{N}$, when generally the angular observations should be repeated.

When the value of e (and hence a) has been assigned, the value of c will be determinate from $E^2 = c^2\Sigma s + a^2\Sigma ms^2$.

Closing errors. The value of E in the foregoing expression also represents the probable error of closure of a traverse. The form differs appreciably from the usual rules for permissible error E_p, which embody the angular term as $a^2N(\Sigma s)^2$, giving an abnormal effect, as though the traverse were a rigid wire with the final angular error eN applied through the entire perimeter $P = \Sigma s$. Nevertheless rules of this nature have proved trust-

* *Engineering*, Aug. 29, 1939.

worthy under a wide range of conditions, and serve as reliable empirical criteria ; *e.g.*

$$E_p = \pm \frac{P}{1000} \sqrt{\left(c^2 + \frac{e^2 N}{12}\right)}; \quad E_p = \pm \sqrt{c^2 P + (0{\cdot}0003 Pe\sqrt{N})^2}.$$

(ii). **Assumed values.** On the other hand, the values of a and c may be assigned in any given case, either by comparative estimate or independent observation ; and from the purely theoretical point of view this is the more satisfactory procedure. Thus e may be taken at the following values for single sights and readings with the following divisions : $1'$ for $1'$, $\frac{3}{4}'$ for $\frac{1}{2}'$, $25''$ to $30''$ for $20''$ verniers, and $15''$ to $20''$ for $10''$ microscopes. Or the probable errors in back angles may be determined by the methods described on pp. 402, 403.

In the simplified methods the values of k may be taken to vary (to a limit of $N = 25$) from 3 to 1 in good compass surveys, $3(6 - N/5)$ in tacheometric traverses, and from $(5 - 0{\cdot}2N)$ to $(1{\cdot}5 - 0{\cdot}06N)$ in theodolite surveys with instruments reading from $1'$ to $20''$.

Example. In the illustrative example on p. 443,

$$\alpha = 3' = e\sqrt{7}, \text{ or, } e = 1{\cdot}134' \text{ with } a = 0{\cdot}3402 \times 10^{-3}.$$

Also $E^2 = 170{\cdot}09 = c^2 \Sigma s^2 + a^2 \Sigma m s^2$, whence $c^2 = 7{\cdot}609 \times 10^{-6}$ from the following tabular values :

Line	*AB*	*BC*	*CD*	*DE*
$s^2 \times 10^6$	9·15	3·70	0·34	1·01
$ms^2 \times 10^6$	9·15	7·41	1·02	4·03
Corr. l :	+3·33	+1·37	+0·13	+0·38
Corr. d :	−4·28	−1·76	−0·16	−0·49
Corr. lat. :	−1219·7	+1511·1	+493·5	+985·8
Corr. dep. :	+2762·2	+1191·9	−311·3	−193·5

Line	*EF*	*FG*	*GA*	Sums
$s^2 \times 10^6$	1·91	0·62	4·60	21·33
$ms^2 \times 10^6$	9·55	3·71	32·18	67·05
Corr. l :	+0·74	+0·24	+1·82	+ 8·01
Corr. d :	−0·95	−0·31	−2·34	−10·29
Corr. lat. :	+43·7	−761·0	−1053·4	− 0·1
Corr. dep. :	−1381·9	−198·8	−1868·7	− 0·1

(In this survey the chaining was necessarily rough, with a ratio of possibly $\frac{1}{700}$; and, though many lengths were laid down, the square root law is inadmissible.)

III. **Proportional co-ordinates.**

The so-called **transit rule** has no rigorous theoretical basis, though many prefer it to the Bowditch method. Here the corrections in latitude and departure are taken in the proportion of the individual latitudes and departures to the *arithmetical sums* of the latitudes $[\lambda]$ and departures $[\delta]$; or

$$l_1 = \frac{\lambda_1}{[\lambda]} E_l; \quad d_1 = \frac{\delta_1}{[\delta]} E_d. \qquad \text{..................(17), (18)}$$

The following notes refer to the adjustment of the traverse of p. 443 by this method, with $[\delta] = 7910{\cdot}1$ and $[\lambda] = 6070{\cdot}8$, and

$$\frac{1}{[\delta]} = \frac{1}{768{\cdot}0} \qquad \frac{1}{[\lambda]} = \frac{1}{758{\cdot}9}.$$

Line	*AB*	*BC*	*CD*	*DE*
Corr. l:	$+1{\cdot}61$	$+1{\cdot}99$	$+0{\cdot}64$	$+1{\cdot}30$
Corr. d:	$-3{\cdot}62$	$-1{\cdot}55$	$-0{\cdot}41$	$-0{\cdot}25$
Corr. lat.:	$-1221{\cdot}4$	$+1511{\cdot}7$	$+494{\cdot}0$	$+986{\cdot}6$
Corr. dep.:	$+2762{\cdot}9$	$+1192{\cdot}1$	$-311{\cdot}5$	$-193{\cdot}2$

Line	*EF*	*FG*	*GA*	Sums
Corr. l:	$+0{\cdot}06$	$+1{\cdot}00$	$+1{\cdot}39$	$+7{\cdot}99$
Corr. d:	$-1{\cdot}80$	$-0{\cdot}26$	$-2{\cdot}43$	$-10{\cdot}32$
Corr. lat.:	$+43{\cdot}1$	$-760{\cdot}2$	$-1053{\cdot}8$	$0{\cdot}0$
Corr. dep.:	$-1382{\cdot}7$	$-198{\cdot}8$	$-1868{\cdot}8$	$0{\cdot}0$

IV. **Orthomorphic methods.**

In these methods it is assumed that the error of closure arises solely from linear sources, and usually any angular error is distributed before the latitudes and departures are calculated.

(*a*) **Crandall's method.** If λ and δ are the respective latitude and departure of any side s, the correction of which is xs, then the respective corrections in latitude and departure will be $l = x\lambda$ and $d = x\delta$, with x varying with different lengths s.

Hence, if linear measurement follows the square root law, the weight will be $1/s$, and if E_l and E_d are respectively the total corrections, then

for $\Sigma \frac{u^2}{s}$ to be a minimum, with

$$E_l = \Sigma x\lambda, \text{......(i)} \quad \text{and} \quad E_d = \Sigma x\delta, \text{......(ii)};$$

$$\Sigma(xs \,.\, dx) = 0, \text{......(iii)}; \quad \Sigma\lambda \,.\, dx = 0, \text{......(iv)}; \quad \text{and} \quad \Sigma\delta \, dx = 0, \text{......(v)}$$

Applying the multipliers $-\eta$ and $-\zeta$ to equations (iv) and (v) respectively, adding equations (iii), (iv), and (v), and equating the coefficients of each dx to zero :

$$x_1 = \frac{\eta\lambda_1 + \zeta\delta_1}{s_1}; \quad x_2 = \frac{\eta\lambda_2 + \zeta\delta_2}{s_2}; \text{ etc. } \text{..................(vi)}$$

Substituting these values of x_1, x_2, etc., in equations (i), (ii),

$$\eta\Sigma\left(\frac{\lambda^2}{s}\right) + \zeta\Sigma\left(\frac{\lambda\delta}{s}\right) = E_l, \text{ ....(vii)}$$

$$\eta\Sigma\left(\frac{\lambda\delta}{s}\right) + \zeta\Sigma\left(\frac{\delta^2}{s}\right) = E_d, \text{ ....(viii)}$$

whence η and ζ are determinate, leading to the following corrections in latitude and departure :

$$l_1 = \eta\,\frac{\lambda_1^2}{s_1} + \zeta\,\frac{\lambda_1\delta_1}{s_1}, \text{ etc. ....(19)}$$

$$d_1 = \eta\,\frac{\lambda_1\delta_1}{s_1} + \zeta\,\frac{\delta_1^2}{s_1}, \text{ etc. ....(20)}$$

If the linear measurements are weighted, w_1, w_2, etc., are merely applied to the corresponding lengths s_1, s_2, etc., in the denominators.

Rappleye's method is a modification of this process, specially applicable to tacheometric traverses.

(*b*) **Ormsby's method.** This simple conventional adjustment may be detailed concisely as follows :

(1) Apply corrective factors to each traverse line thus :
  x for same signs in the individual latitudes and departures.
  y ,, opposite ,, ,, ,, ,, ,, ,,

(2) Select as *primary* co-ordinates those which have the *greater* total correction, E_l or E_d, and apply the sign of that correction to all the individual corrections of the primary co-ordinates, the sign of the correction being opposite to that of the error.

(3) Make the signs of the corrections to the corresponding secondary co-ordinates consistent with these, giving them the $\left\{\begin{matrix}\text{same}\\\text{opposite}\end{matrix}\right\}$ sign $\left\{\begin{matrix}\text{as}\\\text{to}\end{matrix}\right\}$ that of the co-ordinates when the primary co-ordinate and its correction are of the $\left\{\begin{matrix}\text{same}\\\text{opposite}\end{matrix}\right\}$ sign.

Thus, if the primary latitude and its correction are + or −, give the ± sign to the ± secondary departure; but if the primary latitude and its correction are + and −, give the ± sign to the ∓ secondary departure.

(4) Write a pair of simultaneous equations with the above signs, and hence determine the values of x and y, thus:

(5) Finally apply the corrective factors with appropriate signs to the individual latitudes and departures, thus:

$$l_1 = x(\text{or } y)\,\lambda_1, \text{ etc.}; \quad d_1 = x(\text{or } y)\,\delta_1, \text{ etc.}$$

In the following application of this method to the illustrative example of p. 443, the line preceding the observed latitudes shows the designation of the co-ordinates (with primary departures), and the signs succeeding the evaluated co-ordinates indicate the initial signs of the corrections, which latter are finally influenced by the signs of x and y.

Line	*AB*	*BC*	*CD*	*DE*
	y	x	y	y
Obs. lat.:	−1223·0	+1509·7	+493·4	+985·3
Secondary:	+	−	+	+
Obs. dep.:	+2766·5	+1193·7	−311·1	−193·0
Primary:	−	−	−	−
Corr. l:	+3·02	+0·55	+1·22	+2·43
Corr. d:	−6·84	+0·44	−0·77	−0·48
Corr. lat.:	−1220·0	+1510·3	+494·6	+987·7
Corr. dep.:	+2759·7	+1194·1	−311·9	−193·5

Line	*EF*	*FG*	*GA*	Sums
	y	x	x	
Obs. lat.:	+43·0	−761·2	−1055·2	−8·0
Secondary:	+	−	−	
Obs. dep.:	−1380·9	−198·5	−1866·4	+10·3
Primary:	−	−	−	
Corr. l:	+0·11	+0·28	+0·39	+ 8·00
Corr. d:	−3·42	+0·07	+0·69	−10·31
Corr. lat.:	+43·1	−760·9	−1054·8	0·0
Corr. dep.:	−1384·3	−198·4	−1865·7	0·0

Multiplying the initially-prefixed departures through by -1 :

$$+2766{\cdot}5y+1193{\cdot}7x+311{\cdot}1y+193{\cdot}0y+1380{\cdot}9y+198{\cdot}5x+1866{\cdot}4x = +10{\cdot}3$$

$$+1223{\cdot}0y-1509{\cdot}7x+493{\cdot}4y+985{\cdot}3y+\ \ 43{\cdot}0y-761{\cdot}2x-1055{\cdot}2x = +\ 8{\cdot}0$$

$$4651{\cdot}5y+3258{\cdot}6x=+10{\cdot}3. \qquad 2744{\cdot}7y-3326{\cdot}1x=+\ 8{\cdot}0. \qquad \text{Whence } x=-\frac{1}{2730};\ y=+\frac{1}{404{\cdot}8}.$$

V. **Graphical methods.**

Graphical methods should be applied only to low-grade traverses on small scales, more particularly compass surveys, though the process is often admissible to traverses run expressly for contour location.

(*a*) The following application of the Bowditch compass rule is too well known to require more than a brief outline with reference to a traverse originally plotted as *Abcdefga* with a closing error *aA*.

(1) Draw a parallel to the closing line *aA* through the ends of each traverse line. (2) Draw a horizontal base *Oa* equal in length to the total perimeter, and along this base scale off from *O* the respective lengths of the sides *Ab, bc, cd,* etc., on any convenient scale. (3) Erect a perpendicular *aA* at the right-hand extremity of the base equal to the closing error *aA* on the scale of the map, join *OA*, forming a right-angled triangle, and at each of the intermediate points *b, c, d,* etc., erect perpendiculars to determine the corresponding portions of the closing error, *bB, cC, dD*, etc., respectively. (4) Along the parallels through the stations of the traverse skeleton, set off the corresponding parts of the closing error ; *bB* at *b*, *cC* at *c*, *dD* at *d*, and so on ; and connect the points thus determined, obtaining the closed outline *ABCDEFGA*.

The method also applies in adjusting a traverse run between triangulation stations.

(*b*) **Axis method.** The following geometrical construction preserves the shape of the figure by giving all the corrections to the sides without altering the angles.

Consider a six-sided traverse *ABCDEFGA′* lettered clockwise with *EF* horizontal, the figure being uncorrected with a closing error *AA′*.

(1) Join *AA′*, and if this line when produced does not cut the figure into nearly equal parts, transfer the error to some other point say *C′*, by drawing *A′B′* parallel and equal to *AB*, and *B′C′* parallel and equal to *BC*. Then the figure *CDEFA′B′C′* is the uncorrected traverse with *CC′* the closing error, and the line *CC′* produced to *m* in *EF* will presumably divide the polygon into approximately equal parts. Join *A′*, *B′*, *C′*, and *D′* with dotted radials to *m*.

(2) Divide the error *CC′* at *c* in the ratio of the perimeter of one part to the perimeter of the remainder (exclusive of *Cm*), the inner portion *cC′* applying to the portion to the left of *Cm*, thus :

$$\frac{Cc}{\quad} = \frac{CD + DE + Em}{\quad ' + A'B' + B'C'}.$$

(3) Having determined the point c from this relation, plot its position on CC', and through c draw cd parallel to CD, meeting Dm at d, de parallel to DE, meeting mE at e; and so on, the parallels to the traverse lines being intercepted between the radials to m.

QUESTIONS ON ARTICLE 6

1†. Derive from first principles the Bowditch rule for the adjustment of traverse errors, the relevant corrections being in the ratios of the several sides to the perimeter of the polygon. (U.L.)

2†. The errors of latitude and departure in a five-sided traverse were respectively $-1 \cdot 15$ ft. and $-2 \cdot 65$ ft., the lengths of the sides being in order 896, 567, 406, 409, and 352·5 ft.

Derive and employ a formula for the *probable* linear closing error, assuming that the chaining follows a linear law of 1 in 1000 and that the angular error per angle is 1′. State also the ratio of the probable to the actual closing error. (U.L.)

[$E_p = \sqrt{c^2 \Sigma s^2 + a^2 \Sigma m s^2} = 1 \cdot 37$ ft. Actual error, 2·88 ft.; ratio, 1/2·11]

3*. The following co-ordinates were calculated for a closed theodolite traverse:

Line	Lat. (ft.)	Dep. (ft.)
AB	+ 230·6	+ 990·1
BC	− 1663·0	+ 280·5
CD	+ 642·0	− 1799·6
DA	+ 795·1	+ 533·9

Adjust the traverse by any analytical method which involves the assumption that error occurs both in angles and distances.

State clearly why you would not modify the method if the observed angles checked. (U.L.)

4†. A subsidiary traverse of a topographical survey was run between the stations M and N of the triangulation, the following notes being recorded:

Station	Line	Length	Bearing	Total co-ordinates	
		(ft.)		N.	E.
M				2420·0 ft.	2106·4 ft.
,,	$M - 1$	642	N. 26° 15′ E.		
1	1 − 2	540	N. 30° 20′ E.		
2	2 − N	825	N. 63° 45′ E.		
N				3830·0	3408·0

Tabulate the co-ordinates of 1 and 2, making such adjustments as may be necessary. (U.L.)

[Calculated co-ordinates of 1, 2, and *N*, with corrections in brackets:

N. 2995·8 (+1·1); E. 2390·3 (+1·6); N. 3461·9 (+1·9), E. 2663·0 (+3·0); N. 3826·8 (+3·2), E. 3402·9 (+5·1).]

5†. A theodolite traverse *ABCDA* is run between two trigonometrical stations *A* and *D*, the following values of the latitude and departure being found:

Side	Latitude (ft.)	Departure (ft.)
AB	+ 900·3	+ 981·2
BC	− 1876·5	+ 747·0
CD	− 199·7	− 958·1
DA	+ 1172·0	− 768·0

Adjust these values to the nearest tenth of a foot, to the known length of *AD* *without* altering the bearings of the lines. (U.L.)

[Corrected lat.	(Crandall)	+ 900·5,	− 1872·7,	− 199·8.
	(Ormsby)	+ 900·1,	− 1872·3,	− 199·8.
Corrected dep.	(Crandall)	+ 981·5,	+ 745·4,	− 958·9.
	(Ormsby)	+ 981·0,	+ 745·3,	− 958·3.]

6†. The following notes refer to a theodolite-chain traverse of a meadow. Show in supplementary columns the corrected latitudes and departures and the modified bearings when the traverse is adjusted by (1) Bowditch's Rule and (2) Modified Compass Rule, the linear errors following respectively the $\sqrt{s}$ and s laws.

Line	Length	Bearing	Latitude	Departure	Sums
AB	1381·6	S. 88° 13′ E.	− 43·0	+ 1380·9	Σs, 5215 lks.
BC	1064·0	N. 17° 30′ E.	+ 1015·0	+ 319·9	
CD	569·0	N. 16° 19′ W.	+ 546·1	− 159·9	
DE	1026·0	S. 53° 04′ W.	− 616·5	− 820·1	
EF	632·4	S. 29° 14′ W.	− 551·9	− 308·8	E_l, + 2·50
FA	542·0	S. 49° 45′ W.	− 347·2	− 410·1	E_d, + 1·90

[(1) Corr. lat.:

− 43·66 + 1014·49 + 545·83 − 616·99 − 552·20 − 347·46; E_l, + 0·01

Corr. dep.:

+ 1380·40 + 319·51 − 160·11 − 820·47 − 309·03 − 410·30; E_d, 0·00

Mod. bear.:

88° 11½′ 17° 29′ 16° 21′ 53° 03½′ 29° 14′ 49° 44½′

(2) Corr. lat.:

− 43·93 + 1014·45 + 545·94 − 617·02 − 552·10 − 347·46; E_l, − 0·12

Corr. dep. :

$+1380 \cdot 19$ $+319 \cdot 48$ $-160 \cdot 00$ $-821 \cdot 49$ $-308 \cdot 05$ $-410 \cdot 21$; E_d, $-0 \cdot 98$

Mod. bear. :

$88° \ 10\frac{1}{2}'$ $17° \ 29'$ $16° \ 20\frac{1}{4}'$ $53° \ 05\frac{1}{2}'$ $29° \ 13\frac{1}{2}'$ $49° \ 44']$

7†. Calculate the corrections for the above traverse by the Transit Rule, or proportional co-ordinates.

[Lat. corr. l : $+0 \cdot 03$ $0 - \cdot 81$ $-0 \cdot 44$ $+0 \cdot 49$ $+0 \cdot 44$ $+0 \cdot 28$; E_l, $-0 \cdot 01$
Dep. corr. d : $-0 \cdot 77$ $-0 \cdot 18$ $+0 \cdot 09$ $+0 \cdot 46$ $+0 \cdot 17$ $+0 \cdot 23$; E_d, $-0 \cdot 00$]

8. Draw up a list of the sources of error, both instrumental and observational, which affect the accuracy of measurement of the bearings of a theodolite traverse.

The angular work of a closed traverse checked satisfactorily in the field, and the closing error was computed from the coordinates.

State your opinion of the value of this error as an index of the precision of the survey. (I.C.E.)

9. What checks can be applied to a closed traverse survey, and how are residual errors eliminated? Describe briefly the method you consider the most accurate of using a theodolite on a closed traverse survey. State the reason for your selection. (I.C.E.)

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